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Completions of partial metric spaces

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This paper gives the existence and uniqueness theorems in the classical sense for completions of partial metric spaces.

1. Introduction

Partial metric spaces were introduced and investigated by S. Matthews in [10] (also see [3]). In the past years, partial metric spaces had aroused popular attentions and many interesting results are obtained (for example, see [1,2,6,7,9,11]). It is well known that every metric space has a unique completion in the classical sense [5]. But, we do not know if there are similar completion theorems for partial metric spaces. In their paper [8], R. Kopperman, S. Matthews and H. Pajoohesh investigated some notions of completion of partial metric spaces, including the bicompletion, the Smyth completion, and a new "spherical completion". However, completion problem in the classical sense for partial metric spaces is still open. Indeed, it is the most difference from metric that self-distances of some points in partial metric spaces may not be zero, and then $y \in B(x, \varepsilon)$ and $x \in B(y, \varepsilon)$ are not equivalent in general for "ball-neighborhoods" $B(x, \varepsilon)$ and $B(y, \varepsilon)$, which generates many difficulties in investigating partial metric spaces by metric methods.

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In this paper, we introduce symmetrically dense subsets of partial metric spaces to prove the existence and uniqueness theorems in the classical sense for completions of partial metric spaces.

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}^*$ denote the set of all natural numbers and the set of all nonnegative real numbers, respectively.

2. Preliminaries

Definition 1 *([3])*. Let X be a non-empty set. A mapping $p : X \times X \longrightarrow R^*$ is called a partial metric and (X, p) is called a partial metric space if the following are satisfied for all $x, y, z \in X$.

(1) $x = y \Longleftrightarrow p(x, x) = p(y, y) = p(x, y)$.
(2) $p(x, y) = p(y, x)$.
(3) $p(x, x) \leq p(x, y)$.
(4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Remark 1. Let (X, p) be a partial metric space, $x \in X$ and $\varepsilon > 0$. Put $B(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ and put $\mathscr{B} = \{B(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$. Then $\mathscr{B}$ is a base for some topology τ on X ([3]). In this paper, the partial metric space (X, p) is always a topological space (X, τ).

Definition 2 *([3])*. Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ in X is called to be a Cauchy sequence if there is $r \in \mathbb{R}^*$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = r$.

(2) A sequence $\{x_n\}$ in X is called to converge in (X, p) if there is $x \in X$ such that $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$.

(3) (X, p) is called to be complete if every Cauchy sequence in X converges in (X, p).

Remark 2. Let (X, p) be a partial metric space. In this paper, a convergent sequence $\{x_n\}$ in X always means that $\{x_n\}$ converges in (X, p), which is different from that $\{x_n\}$ converges in (X, τ). It is also worth noting that if $\{x_n\}$ converges to x in (X, p), then $\{x_n\}$ converges to x in (X, τ), but the other direction only yields $p(x, x) = \lim_{n\to\infty} p(x, x_n)$ and $\limsup_{n\to\infty} p(x_n, x_n) \leq p(x, x)$.

The following two definitions adopt the descriptions on "isometry" and "dense" for metric case in [5], respectively.

Definition 3. Let (X, p) and (Y, q) be partial metric spaces. A mapping $f : X \longrightarrow Y$ is called to be an isometry if $q(f(x), f(x')) = p(x, x')$ for all $x, x' \in X$.

Definition 4. Let (X, p) be a partial metric space. A complete partial metric space (X^*, p^*) is called a completion of (X, p) if there is an isometry $f : X \longrightarrow X^*$ such that $f(X)$ is dense in (X^*, p^*).

"Sequentially dense" in topological spaces was introduced by S. Davis in [4], we introduce "sequentially dense" in partial metric spaces as follows.

Definition 5. Let (X, p) be a partial metric space and Y be a subset of X.

(1) Y is called to be sequentially dense in X if for any $x \in X$ there is a sequence in Y converging to x.
(2) Y is called to be symmetrically dense in X if for any $x \in X$ and any $\varepsilon > 0$, there is $y \in Y$ such that $y \in B(x, \varepsilon)$ and $x \in B(y, \varepsilon)$.