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Whitney levels in hyperspaces of non-metrizable continua

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ABSTRACT

Let X be a Hausdorff continuum (a nondegenerate, compact, connected, Hausdorff space). Let $C(X)$ (respectively $F_1(X)$) denote the hyperspace of its subcontinua (respectively, its one-point sets), endowed with the Vietoris topology. In this paper we introduce the definition of Whitney levels in $C(X)$ and discuss some basic properties. With this definition, the subsets $F_1(X)$ and $\{X\}$ of $C(X)$ are Whitney levels in $C(X)$, so we call them trivial Whitney levels. In the particular case when X is a generalized arc, we give a condition for the existence of non-trivial Whitney levels in its hyperspace of subcontinua. Finally, we apply this result to the study of Whitney levels in $C(X)$ when X is the Long Arc and the Lexicographic Square.

1. Introduction

A *Hausdorff continuum* is a nondegenerate compact connected Hausdorff space. Given a Hausdorff continuum X we consider the following hyperspaces of X:

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\},$$
$$C(X) = \{A \in 2^X : A \text{ is connected}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}.$$

The hyperspace 2^X is endowed with the Vietoris topology [3, Definition 1.1 and Theorem 1.2].

A *generalized arc* is a Hausdorff continuum X with a linear order and the topology induced by its order. That is, the topology on X is given by the subbasis: $\{(\leftarrow, x) : x \in X\} \cup \{(x, \rightarrow) : x \in X\}$, where $(\leftarrow, x) = \{y \in X : y < x\}$ and $(x, \rightarrow) = \{y \in X : x < y\}$. Every generalized arc X contains a minimal

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element (denoted by $\min(X)$) and a maximal element (denoted by $\max(X)$). An *arc* is a space homeomorphic to the unit interval $[0, 1]$.

Given a hyperspace $H(X)$ of a Hausdorff continuum X, a *Whitney map* for $H(X)$ is a continuous function $\mu : H(X) \to \alpha$, where α is a generalized arc, such that:

(a) $\mu(\{x\}) = \min(\alpha)$ for each $x \in X$, and

(b) if $A, B \in H(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$.

It is usual to find in the literature the definition of Whitney map by taking $\alpha = [0, 1]$. The following question remains open.

Question 1. Does there exist a Hausdorff continuum such that its hyperspace $C(X)$ admits a Whitney map into a generalized arc but it does not admit one into $[0, 1]$?

The following theorem was proved by J.J. Charatonik and W.J. Charatonik in [1, Theorem 1].

Theorem 2. *([1, Theorem 1]) The following conditions are equivalent for a Hausdorff continuum X:*

(a) X is metrizable,

(b) there exists a Whitney map for 2^X, and

(c) there exists a Whitney map for $F_2(X)$.

In the case that X is a metric continuum, Whitney maps for $C(X)$ into $[0, 1]$ can be defined. An example of a non-metrizable continuum such that $C(X)$ admits a Whitney map into $[0, 1]$ is presented in [1].

If X is a generalized arc, then $F_2(X)$ can be embedded in $C(X)$ by assigning to each element $\{x, y\}$ in $F_2(X)$ the minimal subcontinuum containing $\{x, y\}$. Hence, by Theorem 2, if X is a non-metrizable generalized arc, then there are no Whitney maps for $C(X)$.

Given a Hausdorff continuum X and $A, B \in C(X)$ such that $A \subsetneq B$, an *order arc from A to B in $C(X)$* is a subcontinuum $\mathcal{A}$ of $C(X)$ such that for every $C, D \in \mathcal{A}$, $A \subset C \subset B$ and either $C \subset D$ or $D \subset C$. A *long order arc* is an order arc from $\{x\}$ to X, for some $x \in X$.

If X is a **metric** continuum a *Whitney level* in $C(X)$ is a set of the form $\mu^{-1}(t) \subset C(X)$, where $\mu : C(X) \to [0, 1]$ is a Whitney map and $t \in [0, 1]$.

As we mentioned before, when X is a non-metrizable continuum, $C(X)$ does not necessarily admit Whitney maps. In [2, Theorem 1.2], A. Illanes proved that Whitney levels can be defined without using Whitney maps. His result is the following.

Theorem 3. *([2, Theorem 1.2]) Let X be a metric continuum and let $\mathcal{W}$ be a compact subset of $C(X)$ such that $\mathcal{W} \cap F_1(X) = \emptyset$. Then $\mathcal{W}$ is a Whitney level in $C(X)$ if and only if $\mathcal{W}$ satisfies the following two conditions:*

(a) if $A, B \in \mathcal{W}$ and $A \neq B$, then $A \not\subset B$ and $B \not\subset A$,

(b) $\mathcal{W}$ intersects every long order arc in $C(X)$.

The characterization given in Theorem 3 allows us to define Whitney levels for Hausdorff continua X, even if the hyperspace $C(X)$ does not necessarily admit Whitney maps.

Definition 4. Let X be a Hausdorff continuum. Then a *Whitney level in $C(X)$* is a compact subset $\mathcal{W}$ of $C(X)$ such that either $\mathcal{W} = F_1(X)$ or $\mathcal{W} \cap F_1(X) = \emptyset$ and $\mathcal{W}$ satisfies the following two conditions:

(4.1) if $A, B \in \mathcal{W}$ and $A \neq B$, then $A \not\subset B$ and $B \not\subset A$,

(4.2) $\mathcal{W}$ intersects every long order arc in $C(X)$.

Notice that $\{X\}$ and $F_1(X)$ are Whitney levels in $C(X)$ and they are called *trivial Whitney levels*.