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Alternating distances of knots and links

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ABSTRACT

An alternating distance is a link invariant that measures how far away a link is from alternating. We study several alternating distances and demonstrate that there exist families of links for which the difference between certain alternating distances is arbitrarily large. We also show that two alternating distances, the alternation number and the alternating genus, are not comparable.

1. Introduction

Alternating links play an important role in knot theory and 3-manifold geometry and topology. Link invariants are often easier to compute and take on special forms for alternating links. Moreover, the complements of alternating links have interesting topological and geometric structures. Many generalizations of alternating links exist, and a particular generalization can give rise to an invariant that measures how far a link is from alternating. We study several such invariants, which we call alternating distances.

A link L is *split* if it has a separating sphere, i.e. a two-sphere S^2 in S^3 such that L and S^2 are disjoint and each component of $S^3 - S^2$ contains at least one component of L. We will mostly be concerned with non-split links, that is links with no separating spheres. A real valued link invariant $d(L)$ is an *alternating distance* if it satisfies the following conditions.

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(1) For any non-split link, $d(L) \geq 0$.
(2) For any non-split link, $d(L) = 0$ if and only if L is alternating.
(3) If L_1 and L_2 are non-split links and $L_1 \# L_2$ is any connected sum of L_1 and L_2, then $d(L_1 \# L_2) \leq d(L_1) + d(L_2)$.

The connected sum $L_1 \# L_2$ depends on a choice of components in L_1 and L_2. However, condition (3) above must be true for any choice of connected sum. We will frequently use the notation $L_1 \# L_2$ to denote an arbitrary choice of connected sum.

We consider the following invariants. The dealternating number, denoted by $\mathrm{dalt}(L)$, and the alternation number, denoted by $\mathrm{alt}(L)$, are defined by counting crossing changes. The relationship between the minimum crossing number $c(L)$ of a link L and the span of the Jones polynomial $V_L(t)$ of L was used to prove some of Tait's famous conjectures, and we study the difference $c(L) - \mathrm{span}\, V_L(t)$. The Turaev genus, denoted by $g_T(L)$, and the alternating genus, denoted by $g_{\mathrm{alt}}(L)$, are the genera of certain surfaces associated to L. The warping span, denoted by $\mathrm{warp}(K)$, is defined by examining the over-under behavior as one travels along the knot or link. Precise definitions of these invariants are given in Section 2.

Let d_1 and d_2 be real valued link invariants and let $\mathcal{F}$ be a family of links. We say d_2 *dominates* d_1 on $\mathcal{F}$, and write $d_1(\mathcal{F}) \ll d_2(\mathcal{F})$, if for each positive integer n, there exists a link $L_n \in \mathcal{F}$ such that $d_2(L_n) - d_1(L_n) \geq n$.

In Section 4, we examine three families of links. The first family $\mathcal{F}(W_n)$ consists of iterated Whitehead doubles of the figure-eight knot. The second family $\mathcal{F}(\widetilde{T}(p,q))$ consists of links obtained by changing certain crossings of torus links. The third family $\mathcal{F}(T(3,q))$ consists of the $(3,q)$-torus knots.

Theorem 1.1. *Let $\mathcal{F}(W_n)$, $\mathcal{F}(\widetilde{T}(p,q))$, and $\mathcal{F}(T(3,q))$ be the families of links above.*

(1) *The dealternating number, $c(L) - \mathrm{span}\, V_L(t)$, and Turaev genus dominate the alternation number on $\mathcal{F}(W_n)$.*
(2) *The dealternating number, Turaev genus, $c(L) - \mathrm{span}\, V_L(t)$, and alternation number dominate the alternating genus on $\mathcal{F}(\widetilde{T}(p,q))$.*
(3) *The dealternating number, alternation number, $c(L) - \mathrm{span}\, V_L(t)$, and Turaev genus dominate the warping span on $\mathcal{F}(T(3,q))$.*
(4) *The difference $c(L) - \mathrm{span}\, V_L(t)$ dominates the dealternating number, alternation number, alternating genus, and Turaev genus on $\mathcal{F}(T(3,q))$.*

Two real valued link invariants d_1 and d_2 are said to be *comparable* if either $d_1(L) \leq d_2(L)$ or $d_2(L) \leq d_1(L)$ for all links L. The invariants d_1 and d_2 are not comparable if there exist links L and L' such that $d_1(L) < d_2(L)$ and $d_1(L') > d_2(L')$.

Theorem 1.2. *The alternation number and alternating genus of a link are not comparable.*

This paper is organized as follows. In Section 2, we define the invariants mentioned in Theorems 1.1 and 1.2. In Section 3, we describe some lower bounds for the invariants. In Section 4, we define several families of links and use them to prove Theorems 1.1 and 1.2. In Section 5, we discuss some open questions about our invariants.

2. The invariants

In this section, the invariants of Theorems 1.1 and 1.2 are defined. We show that each one is an alternating distance, and discuss some known relationships between them.