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Separation of analytic sets by rectangles of low complexity

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We provide Hurewicz tests for the separation of disjoint analytic sets by rectangles of the form $\Gamma \times \Gamma'$ for $\Gamma, \Gamma' \in \{\boldsymbol{\Sigma}^0_1, \boldsymbol{\Pi}^0_1, \boldsymbol{\Pi}^0_2\}$.

1. Introduction

One of the turning points in Descriptive Set Theory was the realization of the fact that the continuous images of Borel sets are not necessarily Borel, but that they define a new class, that of the analytic sets. However, this class kept some of the nice structural properties of other well known classes.

A remarkable example was Lusin's separation theorem (see for example [3] for this theorem, basic theory and notation), which states that for any two disjoint analytic subsets A and B of a Polish space, one can find a Borel set that contains A and does not intersect B (i.e., separates A from B). A nice refinement was offered by A. Louveau and J. Saint-Raymond in [9], where they gave a test to recognize when two disjoint analytic sets can be separated by a set of a given Borel class.

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A different but related question was first answered by A.S. Kechris, S. Solecki and S. Todorcevic in [4]: Given an analytic graph G (i.e., a symmetric irreflexive relation) on X, when does this graph have a countable Borel coloring, i.e., a Borel function from X to ω whose inverse images of points are G-discrete? This question also makes sense for directed graphs, i.e., irreflexive relations.

We can view this as a separation question on $X \times X$: we are trying to separate the diagonal from G by a union of countably many disjoint Borel "squares" (i.e., sets of the form $C \times C$). The more general problem of separation of analytic sets by a countable union of Borel rectangles (i.e., sets of the form $C \times D$) was in some way treated by J.H. Silver (see for example, [1]). Similar questions were then answered by L.A. Harrington, A.S. Kechris and A. Louveau in [2] and by D. Lecomte in [5].

All these results have something in common. They are what is commonly called a Hurewicz-like test. This consists of an example which is usually simple to understand and which in someway embeds into the sets which do not satisfy a certain property. So for example, in Hurewicz's original theorem, we obtain a set that continuously embeds in all analytic sets which are not $\boldsymbol{\Sigma}^0_2$ sets.

In [5] and [6], D. Lecomte studies the separation of analytic sets by a countable union of Borel rectangles. In order to do this, he introduces the following quasi-order. He finds minimal examples for this quasi-order for pairs of sets without this property.

Definition. For $e \in 2$, let X_e, Y_e be Polish spaces, and $A_e, B_e \subseteq X_e \times Y_e$. We say that (X_0, Y_0, A_0, B_0) reduces to (X_1, Y_1, A_1, B_1) if there are continuous functions $f : X_0 \to X_1$ and $g : Y_0 \to Y_1$ such that

$$A_0 \subseteq (f \times g)^{-1}(A_1)$$

and

$$B_0 \subseteq (f \times g)^{-1}(B_1).$$

In this case, we write $(X_0, Y_0, A_0, B_0) \leq (X_1, Y_1, A_1, B_1)$.

Later, D. Lecomte and M. Zeleny studied in [7] how to solve a different question using this same quasi-order. The problem is to characterize when an analytic set is separable from another one by a countable union of sets of fixed Borel complexity. They also studied the problem of characterizing when an analytic digraph has a coloring of bounded Borel complexity. In particular, they proved the following conjecture (also proposed in [7]) when $\xi = 1, 2$.

Conjecture 1.1. *([7]) For each $0 < \xi < \omega_1$ there are Polish spaces $\mathbb{X}_\xi, \mathbb{Y}_\xi$ and analytic subsets $\mathbb{A}_\xi, \mathbb{B}_\xi$ of $\mathbb{X}_\xi \times \mathbb{Y}_\xi$ such that for all Polish spaces X, Y and analytic disjoint $A, B \subseteq X \times Y$, exactly one of the following holds:*

1. *B is separable from A by a $(\boldsymbol{\Sigma}^0_\xi \times \boldsymbol{\Sigma}^0_\xi)_\sigma$ set,*
2. *$(\mathbb{X}_\xi, \mathbb{Y}_\xi, \mathbb{B}_\xi, \mathbb{A}_\xi) \leq (X, Y, B, A)$.*

We clarify some notation: given a class of sets Γ, as usual $\Gamma(X) = \{A \in \Gamma | A \subseteq X\}$. Also $\Gamma \times \Gamma' = \{A \times B | A \in \Gamma, B \in \Gamma'\}$, and finally $\Gamma_\sigma = \{\bigcup_{n\in\omega} A_n | A_n \in \Gamma\}$. We also use Π_0 and Π_1 for the projection on the first and second coordinate respectively.

How can we characterize the separability by a Borel rectangle? This is probably folklore, but we will provide a proof later on.

Proposition 1.2. *Let X, Y be Polish spaces, and let A, B be $\boldsymbol{\Sigma}^1_1$ subsets of $X \times Y$. The following are equivalent:*