Topology and its Applications 184 (2015) 29–40

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Convergence of sequences of inverse limits

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A R T I C L E I N F O

Article history:
Received 1 April 2014
Received in revised form 21 December 2014
Accepted 22 January 2015
Available online 2 February 2015

MSC:
54F15
54D80
54C60

Keywords:
Inverse limit
Upper semi-continuous
Set-valued functions

A B S T R A C T

We consider the following question. Let X be a compact metric space, and let $(f_n : X \to 2^X)_{n=1}^{\infty}$ be a sequence of upper semi-continuous set-valued functions whose graphs converge to the graph of a function $f : X \to 2^X$ in the hyperspace $2^{X\times X}$. Under what additional assumptions does it follow that the corresponding sequence of inverse limits $(\varprojlim \mathbf{f_n})_{n=1}^{\infty}$ converges to $\varprojlim \mathbf{f}$ in the hyperspace $2^{\prod_{i=1}^{\infty} X}$? We give two nonequivalent conditions which generalize previous answers given by Banič, Črepnjak, Merhar, and Milutinović in 2010 and 2011.

1. Introduction

We will examine the question below which Banič, Črepnjak, Merhar, and Milutinović considered in [1,2]. The following notation will enable the question to be stated more succinctly.

Let X, be a compact metric space, and let $\mathscr{X}$ represent the product space $\prod_{i=1}^{\infty} X$. Given an upper semi-continuous function $f : X \to 2^X$ and a sequence $(f_n : X \to 2^X)_{n=1}^{\infty}$ of upper semi-continuous functions, let $K = \varprojlim \mathbf{f}$ and $K_n = \varprojlim \mathbf{f_n}$ for each $n \in \mathbb{N}$. Additionally, let $\Gamma(f) = \{(x, y) \in X \times X : y \in f(x)\}$, and likewise for each function f_n.

Question 1.1. If $\lim_n \Gamma(f_n) = \Gamma(f)$ in the hyperspace $2^{X\times X}$, under what additional assumptions does it follow that $\lim_n K_n = K$ in the hyperspace $2^{\mathscr{X}}$?

Banič, Črepnjak, Merhar, and Milutinović gave a partial answer in [2, Theorem 3.3], stating that the statement $\lim_n K_n = K$ holds so long as f is continuous and single-valued (i.e. $f : X \to X$), and $\pi_1(K) \subseteq$

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http://dx.doi.org/10.1016/j.topol.2015.01.012
0166-8641/

$\liminf_n \pi_1(K_n)$. We demonstrate that the condition that f be continuous and single-valued may be relaxed in two ways yielding the following theorems.

Theorem 1.2. *Let X be a compact metric space and $f : X \to 2^X$ be upper semi-continuous. For each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be an upper semi-continuous function such that $\lim_n \Gamma(f_n) = \Gamma(f)$ in $2^{X\times X}$. If $\pi_1(K) \subseteq \liminf_n \pi_1(K_n)$ and K has the weak compact full projection property, then $\lim_n K_n = K$ in $2^{\mathscr{X}}$.*

Theorem 1.3. *Let X be a compact metric space and $f : X \to 2^X$ be continuous. For each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be upper semi-continuous. If $\pi_1(K) \subseteq \liminf_n \pi_1(K_n)$, and there exists a set $A \subseteq X$ such that*

(1) *A is dense in $\pi_1(K)$,*
(2) *for each $a \in A$, $A \cap f(a)$ is dense in $f(a)$,*
(3) *$A \subseteq f(A)$, and*
(4) *for each $a \in A$, $(f_n)_{n=1}^{\infty}$ converges uniformly to f on a neighborhood of a,*

then $\lim_n K_n = K$ in $2^{\mathscr{X}}$.

Theorem 1.2 is proven in Section 3, and Theorem 1.3 in Section 4. Additionally, it will be shown that both of these theorems are generalizations of the result of Banič et al. In Section 5, Theorems 1.2 and 1.3 are generalized further to be applicable to non-constant inverse sequences.

Finally, a few examples are given in Section 6.

2. Preliminaries

If X is a compact metric space, we denote by 2^X the set of all non-empty compact subsets of X. We give this set the topology induced by the Hausdorff metric which we now define.

Definition 2.1. Suppose X is a compact metric space with metric d. If $A \subseteq X$ is non-empty and closed, and $\epsilon > 0$, then

$$N(A,\epsilon) = \{x \in X : d(x,a) < \epsilon \text{ for some } a \in A\}.$$

The *Hausdorff metric* $\mathcal{H}_d$ on 2^X is defined by

$$\mathcal{H}_d(A,B) = \inf\{\epsilon > 0 : A \subseteq N(B,\epsilon), \text{ and } B \subseteq N(A,\epsilon)\}.$$

The topological space $(2^X, \mathcal{H}_d)$ is referred to as a *hyperspace* of X.

If X and Y are compact metric spaces and $x \in X$, a function $f : X \to 2^Y$ is said to be *upper semi-continuous* at x if for every open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(t) \subseteq V$ for all $t \in U$. The function f is said to be *upper semi-continuous* if it is upper semi-continuous at each point of X. The *graph* of a function $f : X \to 2^Y$, denoted $\Gamma(f)$, is the subset of $X \times Y$ consisting of all points (x,y) such that $y \in f(x)$. In [6], it was shown that if X and Y are compact metric spaces, a function $f : X \to 2^Y$ is upper semi-continuous if and only if $\Gamma(f)$ is compact.

Suppose $\mathbf{X} = (X_i)_{i\in\mathbb{N}}$ is a sequence of compact metric spaces, and $\mathbf{f} = (f_i)_{i\in\mathbb{N}}$ is a sequence of upper semi-continuous functions such that for each $i \in \mathbb{N}$, $f_i : X_{i+1} \to 2^{X_i}$. Then the pair $\{\mathbf{X}, \mathbf{f}\}$ is called an *inverse sequence*, and the *inverse limit* of that inverse sequence, denoted $\varprojlim \mathbf{f}$, is the set

$$\varprojlim \mathbf{f} = \left\{ \mathbf{x} \in \prod_{i=1}^{\infty} X_i : x_i \in f_i(x_{i+1}) \text{ for all } i \in \mathbb{N} \right\}.$$