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A subsequence theorem for generalised inverse limits

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ABSTRACT

In this paper we give a subsequence theorem for generalised inverse limits. Suppose that for each $i \in \mathbb{N}$, X_i is a compact Hausdorff space and $f_{i+1} : X_{i+1} \to X_i$ is a continuous function. The subsequence theorem for inverse limits states that if $\langle s_i : i \in \mathbb{N}\rangle$ is a strictly increasing sequence of positive integers where $s_0 = 1$, and if $Y_0 = X_0$, for each positive integer i, $Y_i = X_{s_i}$, and for each $i \in \mathbb{N}$, $g_{i+1} = f_{s_i} \circ \cdots \circ f_{s_{i+1}}$, then $\varprojlim(X_i, f_i) = \varprojlim(Y_i, g_i)$. The subsequence theorem does not hold for generalised inverse limits. We give a class of upper semicontinuous set-valued bonding functions for which the theorem does hold.

1. Introduction

Inverse limits are topological spaces that are determined by an inverse sequence of continuous functions (called bonding maps). They have been extensively studied and a great deal is known about them. One central fact about inverse limits is the so-called subsequence theorem, which ensures that the homeomorphism type of an inverse limit is not affected when we “collapse” portions of the inverse sequence by function composition.

More recently, the notion of generalised inverse limits was introduced in [11]. The generalisation is obtained by allowing the bonding maps to be upper semicontinuous and set-valued. Many properties of inverse limits do not hold in general when the bonding maps are set-valued. Recent work by Ingram and Mahavier [6,7,9] and [10] has engendered much interest in the subject with an increasing number of papers appearing every year. For examples see [1–5,8,12].

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In this generalised setting, we do not have a direct generalisation of the subsequence theorem. The question of which sets of conditions on a generalised inverse sequence ensure that an analogue of the subsequence theorem holds in the generalised context was raised by Ingram [7, Problem 6.8]. We give conditions on the bonding maps for which an analogue of the subsequence theorem holds.

We prove the following theorem (the terms and notation are defined in Section 2).

Theorem 1.1. *Suppose that X is a compact Hausdorff space, $D \subset X$, $f_1, \ldots, f_n$ is an (X, D)-collection, $n \geq 1$, and $f_0 : X \to X$ is the identity function. Define the upper semicontinuous set-valued function $F : X \to X$ by*

$$\mathit{\Gamma}(F) := \mathit{\Gamma}(f_0) \cup \cdots \cup \mathit{\Gamma}(f_n).$$

Then the generalised inverse sequence

$$\boldsymbol{F} = X \xleftarrow{F} X \xleftarrow{F} X \xleftarrow{F} \cdots$$

has the collapsing subsequence property.

In Section 2 we give definitions and set up the notation required to state and prove our results. In Section 3 we first develop notation specific to a given system, we then prove a number of technical lemmas and finally we prove the main theorem.

2. Preliminaries

This section provides terminology and notation required to develop the theory behind our main theorem.

Definition 2.1. An *inverse sequence* $\boldsymbol{f}$ consists of a sequence $\langle X_i : i \in \mathbb{N}\rangle$ of compact Hausdorff spaces, and a sequence $\langle f_i : i \in \mathbb{N} \setminus \{0\}\rangle$ of continuous functions, called *bonding maps*, where $f_{i+1} : X_{i+1} \to X_i$ for each $i \in \mathbb{N}$. We denote $\boldsymbol{f}$ as follows:

$$\boldsymbol{f} = X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} \cdots.$$

We denote a point $(x_i) \in \prod_{i\in\mathbb{N}} X_i$ by $\mathbf{x}$, for example.

Definition 2.2. Let

$$\boldsymbol{f} = X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} \cdots$$

be an inverse sequence. The *inverse limit* of $\boldsymbol{f}$ is the space

$$\varprojlim(\boldsymbol{f}) := \left\{\boldsymbol{x} \in \prod_{i\in\mathbb{N}} X_i : \forall i \in \mathbb{N},\ x_i = f_{i+1}(x_{i+1})\right\}$$

considered as a subspace of the product space $\prod_{i\in\mathbb{N}} X_i$.

Definition 2.3. Suppose

$$\boldsymbol{f} = X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} \cdots$$