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On a conjecture by Kauffman on alternative and pseudoalternating links

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ABSTRACT

It is known that alternative links are pseudoalternating. In 1983 Louis Kauffman conjectured that both classes are identical. In this paper we prove that Kauffman Conjecture holds for those links whose first Betti number is at most 2. However, it is not true in general when this value increases, as we also prove by finding two counterexamples: a link and a knot whose first Betti numbers equal 3 and 4, respectively. In the way we work with the intermediate family of homogeneous links, introduced by Peter Cromwell.

1. Introduction

In [7] Louis Kauffman defined the family of alternative links, as an extension of the class of alternating links which preserves some of the nice properties of this well-known class. Previously, in 1976, E.J. Mayland and K. Murasugi introduced the class of pseudoalternating links [8]. Alternative links are pseudoalternating, and Kauffman conjectured that the converse also holds:

Conjecture 1.1. *([7]) The classes of alternative and pseudoalternating links are identical.*

Although this conjecture was stated by Kauffman, Mayland and Murasugi posed a similar question in [8]. In this paper we prove Conjecture 1.1 for links having their first Betti number, β, smaller than 3 (this includes the class of knots of genus one), and we provide counterexamples for links whose first Betti numbers equal 3 and 4, respectively (recall that $\beta(L) = \min\{rank(H_1(F))|$ F is a Seifert surface for L$\}$). Namely, we present a genus two knot and a genus one two-components link which are pseudoalternating but not alternative.

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The plan of the paper is as follows. In Section 2 we recall the definitions of alternative and pseudoalternating links; we also recall the definition of homogeneous links, an intermediate family introduced by Peter Cromwell in [3]. In Section 3 we disprove Kauffman Conjecture by finding a link and a knot being pseudoalternating and non-homogeneous, hence non-alternative: $L9n18\{1\}$ and 10_{145}, whose first Betti numbers equal 3 and 4, respectively. Finally, Section 4 is devoted to prove that the Conjecture holds in the case of links having first Betti number smaller than 3, providing an alternative proof of the characterization of homogeneous genus one knots given in [6].

2. Alternative, homogeneous and pseudoalternating links

As the alternative, homogeneous and pseudoalternating characters of a link are orientation dependent, from now on all links will be oriented and non-split.

Given an oriented diagram D of a link L, it is possible to smooth every crossing coherently with the orientation of the diagram. After doing this for all crossings in D, we obtain a set of topological (Seifert) circles. Following Kauffman, the *spaces* of the diagram D are the connected components of the complement of its Seifert circles in S^2, as opposed to the regions of the knot diagram. Draw an edge joining two Seifert circles at the place where there was a crossing in D, and label the edge with the sign of the corresponding crossing ($+$ or $-$). We will refer to the resulting set of topological circles and labeled edges as the Seifert diagram of D, because of the analogy of this process to Seifert's algorithm for constructing an orientable surface spanning a link.

Definition 2.1. ([7]) An oriented diagram D is alternative if all the edges in any given space of D have the same sign. An oriented link is alternative if it admits an alternative diagram.

An oriented diagram is alternating if and only if it is alternative and the sign of the edges in its Seifert diagram changes alternatively when passing through adjacent spaces [7, Lemma 9.2]. There are nevertheless alternative links which are not alternating: for instance, every positive (hence alternative) non-alternating link, like the knot 8_{19}.

We consider now the family of homogeneous links, introduced by Peter Cromwell in 1989 [3]. From the Seifert diagram associated to D, we can construct a graph G_D as follows: associate a vertex to each Seifert circle and draw an edge connecting two vertices in G_D for each edge joining the associated circles in the Seifert diagram; each edge must be labeled with the sign $+$ or $-$ of its associated crossing in D. The signed graph G_D is called the Seifert graph associated to D. Note that G_D can be obtained from the Seifert diagram of D by collapsing each circle to a vertex.

Given a connected graph G, a vertex v is a *cut vertex* if $G\backslash\{v\}$ is disconnected. A block of G is a maximal subgraph of G with no cut vertices. Blocks of the graph G can be thought of in the following way: remove all the cut vertices of G; each remaining connected component together with its adjacent cut vertices is a block of G.

Definition 2.2. ([3]) A Seifert graph is homogeneous if all the edges of a block have the same sign, for all blocks in the graph. An oriented diagram D is homogeneous if its associated Seifert graph G_D is homogeneous. An oriented link is homogeneous if it admits a homogeneous diagram.

Note that the original diagram D can be recovered from its Seifert diagram, as the sign and position of the crossings in the diagram are preserved (see Fig. 1). However, as the relative position of the circles and the order of the edges is not encoded in the Seifert graph, D cannot be recovered from G_D.

Let us finally introduce pseudoalternating links. Starting from an oriented diagram D of a link L, the surface S_D obtained by applying Seifert's algorithm [4] is known as the canonical surface (called projection