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Period sets of linear toral endomorphisms on $\mathbb{T}^2$

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The period set of a dynamical system is defined as the subset of all integers n such that the system has a periodic orbit of length n. Based on known results on the intersection of period sets of torus maps within a homotopy class, we give a complete classification of the period sets of (not necessarily invertible) toral endomorphisms on the 2-dimensional torus $\mathbb{T}^2$.

1. Introduction

The set of periods that are present in a dynamical system is one of the key quantities which characterise the system. For continuous maps on the interval, the celebrated theorem by Sharkovsky provides a complete characterisation of period sets in terms of the Sharkovsky ordering. A generalisation of Sharkovsky's Theorem to the circle was obtained by the authors Block, Coppel, Guckenheimer, Misiurewicz and Young [1–4], for a unified proof see [5]. For other classes of maps where the period sets have been studied see for instance [6–9].

Toral or torus endomorphisms are continuous mappings of the torus that preserve its group structure, hence in the additive notation $\mathbb{T}^m \cong \mathbb{R}^m/\mathbb{Z}^m$, they can be represented as $m \times m$ integer matrices, see for instance [10, Chapter 0]. In the present article, the term 'toral endomorphism' is always used in this sense, that is, for a map on the torus which is induced by the action of an integer matrix modulo 1. They serve as a standard example in the theory of discrete dynamical systems and ergodic theory, and particularly

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Table 1
Period sets for circle maps.

d	$\mathrm{MPer}(f_A)$	$\mathrm{Per}(f_A)$	$\in \mathrm{Aut}(\mathbb{T})$?
1	$\emptyset$	$\{1\}$	yes
0	$\{1\}$	$\{1\}$	no
-1	$\{1\}$	$\{1,2\}$	yes
-2	$\mathbb{N}\setminus\{2\}$	$\mathbb{N}\setminus\{2\}$	no
$d \in \mathbb{Z}\setminus\{-2,-1,0,1\}$	$\mathbb{N}$	$\mathbb{N}$	no

the case of hyperbolic toral automorphisms, corresponding to integer matrices with determinant ± 1 and no eigenvalues on the unit circle, has been studied extensively because of its interesting dynamical properties, compare [10,11]. In [12], the period sets of toral automorphisms on $\mathbb{T}^2$ were investigated.

Minimal period sets are the intersection of all period sets arising from maps in the same homotopy class; the origins of their study go back to Alsedà, Baldwin, Llibre, Swanson and Szlenk [13], see also [14, 15]. Associated with each homotopy class, one has a unique integer matrix A, which defines the action on the first homology group, and this integer matrix, in turn, defines an endomorphism of the torus, f_A, hence it is itself a member of the homotopy class. An important result in arbitrary dimension is that the minimal period set of a homotopy class "essentially" coincides with the period set of the associated toral endomorphism, apart from possibly those periods that may arise from roots of unity among the eigenvalues of this matrix, see [15]. However, in many of the cases in which ± 1 is among the eigenvalues, one finds that the minimal period set and the period set of the endomorphism totally differ and it can be concluded that, in these cases, the endomorphism is not a good model for the dynamics of its homotopy class with respect to periodic orbits.

For circle maps, the situation is comparatively simple and is displayed in Table 1. The period sets only depend on the degree d of the map, listed in the first column; the second and the third column refer to the minimal period set and the period set of the corresponding endomorphism, respectively; the last column answers the question whether the endomorphism defining the class under consideration is invertible, hence an automorphism. Here, the linear endomorphism f_A is given by $f_{(d)}(x) = d \cdot x$.

In this article, we compare the minimal period sets of maps on $\mathbb{T}^2$ with the period set of the associated toral endomorphism, aiming for an analogue of the above table for dimension 2.

To obtain a complete classification, a variety of partly very different techniques is employed. The minimal period sets were derived in [13] by means of estimating Nielsen numbers, see [7] for background reading on this approach. The study of period sets in the special case of toral automorphisms in [12] is based on an extensive distinction of cases; we note that much of the reasoning in [12] makes use of the assumption of a determinant ± 1 and hence does not directly generalise to arbitrary 2×2 matrices. We complete the classification of period sets by making use of results on *local conjugacy*, that is conjugacy modulo n, $n \in \mathbb{N}$, which corresponds to the action of the matrix on the invariant subsets of points with rational coordinates.

The outline of this article is as follows. In Section 2, we compile the theory of minimal period sets and periods of toral endomorphisms as far as needed for our purpose, derive the period sets $\mathrm{Per}(f_A)$ from the minimal period sets $\mathrm{MPer}(f_A)$ wherever possible and identify the cases that require individual treatment. In Section 3, we complete the classification of period sets on the two-dimensional torus by considering normal forms for matrices with an eigenvalue ± 1. At the end of this introductory section, we summarise the results of our later analysis in form of a comprehensive table.

The following Table 2 constitutes the 2-dimensional analogue of Table 1 for circle maps. Starting from some $A \in \mathrm{Mat}(2,\mathbb{Z})$, the columns (from left to right order) refer to the eigenvalues of A, the pair of the trace and the determinant of A, i.e. $(t,d) = (\mathrm{tr}(A), \det(A))$, the minimal polynomial μ_A, the minimal period set $\mathrm{MPer}(f_A)$ of the homotopy class of f_A, the period set $\mathrm{Per}(f_A)$, and finally an answer to the question whether $f_A \in \mathrm{End}(\mathbb{T}^2)$ is invertible, hence an element of $\mathrm{Aut}(\mathbb{T}^2) \subset \mathrm{End}(\mathbb{T}^2)$. The symbol χ_A denotes