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Bounds for fixed points on hyperbolic manifolds

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A B S T R A C T

For a compact (without boundary) hyperbolic n-manifold M with $n \geq 4$, we show that there exists a finite bound $\mathcal{B}$ such that for any homeomorphism $f : M \to M$ and any fixed point class $\mathbf{F}$ of f, the index $|\mathrm{ind}(f, \mathbf{F})| \leq \mathcal{B}$, which is a partial positive answer of a question given by Jiang in [3]. Moreover, when M is a compact hyperbolic 4-manifold, or a compact hyperbolic n-manifold ($n \geq 5$) with isometry group $\mathrm{Isom}(M)$ a p-group, we give some explicit descriptions of the bound $\mathcal{B}$.

1. Introduction

Fixed point theory studies fixed points of a selfmap f of a space X. Nielsen fixed point theory, in particular, is concerned with the properties of the fixed point set

$$\mathrm{Fix} f := \{x \in X | f(x) = x\}$$

that are invariant under homotopy of the map f (see [2] for an introduction).

The fixed point set $\mathrm{Fix} f$ splits into a disjoint union of *fixed point classes*: two fixed points are in the same class if and only if they can be joined by a *Nielsen path* which is a path homotopic (relative to endpoints) to its own f-image. For each fixed point class $\mathbf{F}$, a homotopy invariant *index* $\mathrm{ind}(f, \mathbf{F}) \in \mathbb{Z}$ is defined. A fixed point class is *essential* if its index is non-zero. The number of essential fixed point classes of f is called the *Nielsen number* of f, denoted by $N(f)$. The famous Lefschetz fixed point theorem says that the sum of the

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indices of the fixed points of f is equal to the *Lefschetz number* $L(f)$, which is defined as

$$L(f) := \sum_q (-1)^q \mathrm{Trace}(f_* : H_q(X; \mathbb{Q}) \to H_q(X; \mathbb{Q})).$$

A compact polyhedron X is said to have the *Bounded Index Property (BIP)* if there is an integer $\mathcal{B} > 0$ such that for any map $f : X \to X$ and any fixed point class $\mathbf{F}$ of f, the index $|\mathrm{ind}(f, \mathbf{F})| \leq \mathcal{B}$. X has the *Bounded Index Property for Homeomorphisms (BIPH)* if there is such a bound for all homeomorphisms $f : X \to X$.

In [3], B. Jiang showed that graphs and surfaces with negative Euler characteristic have BIP (see [5] for an enhanced version), and he asked the following question (see [3, Question 3]).

Question. Does every compact aspherical polyhedron X (i.e. $\pi_i(X) = 0$ for all $i > 1$) have BIP or BIPH?

In [6], C. McCord showed that infrasolvmanifolds (manifolds which admit a finite cover by a compact solvmanifold) have BIP. In [4], B. Jiang and S. Wang showed that geometric 3-manifolds have BIPH for orientation-preserving self-homeomorphisms: the index of each essential fixed point class is ± 1. In [10], the author showed that orientable compact Seifert 3-manifolds with hyperbolic orbifolds have BIPH. Later in [11], the author showed that compact hyperbolic 3-manifolds (not necessarily orientable) also have BIPH.

In this note, we consider fixed point classes of homeomorphisms of compact hyperbolic n-manifolds. By a *compact hyperbolic n-manifold* we mean a quotient space $M = \mathbb{H}^n/\Gamma$, where $\mathbb{H}^n$ is the hyperbolic n-space, that is, the connected, simply connected Riemanian manifold of constant curvature -1, and Γ is a cocompact torsion-free discrete subgroup of the group $\mathrm{Isom}(\mathbb{H}^n)$ of all the isometries of $\mathbb{H}^n$. It is well-known that the isometry group $\mathrm{Isom}(M)$ of a compact hyperbolic n-manifold M of $n \geq 2$ is finite. In fact, for every $n \geq 2$ and every finite group G, there exist infinitely many compact n-dimensional hyperbolic manifolds M with $\mathrm{Isom}(M) \cong G$ (see [1]).

Notice that a compact hyperbolic manifold has empty boundary in this note.

The main result of this note is the following

Theorem 1.1. *Any compact hyperbolic n-manifold with $n \geq 2$ has BIPH.*

Moreover, for any compact hyperbolic n-manifold M ($n \geq 3$) with the isometry group $\mathrm{Isom}(M)$ isomorphic to a p-group (i.e., $|\mathrm{Isom}(M)|$ is a power of some prime p), we have an explicit bound.

Theorem 1.2. *Let M be a compact hyperbolic n-manifold with $n \geq 3$. If the isometry group* $\mathrm{Isom}(M)$ *is a p-group, then for any homeomorphism $f : M \to M$, we have*

$$\max\{N(f), |L(f)|\} \leq \sum_{\mathbf{F} \in \mathrm{Fpc}(f)} |\mathrm{ind}(f, \mathbf{F})| \leq \dim H_*(M; \mathbb{Z}_p),$$

where $\mathrm{Fpc}(f)$ *denotes the set of all the fixed point classes of f, and* $\dim H_*(M; \mathbb{Z}_p)$ *denotes the dimension of the $\mathbb{Z}_p$-linear space $H_*(M; \mathbb{Z}_p) = \bigcup_{r \geq 0} H_r(M; \mathbb{Z}_p)$.*

For any compact hyperbolic 4-manifold, we have

Theorem 1.3. *Let M be a compact hyperbolic* 4*-manifold. Then for any homeomorphism $f : M \to M$, we have*

$$\max\{N(f), |L(f)|\} \leq \sum_{\mathbf{F} \in \mathrm{Fpc}(f)} |\mathrm{ind}(f, \mathbf{F})| \leq \mathcal{B}(M),$$