Topology and its Applications 185–186 (2015) 88–92

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Non-orientable genus of knots in punctured Spin 4-manifolds

Kouki Sato

ARTICLE INFO

Article history:
Received 2 December 2014
Accepted 20 February 2015
Available online 5 March 2015

Keywords:
Knot
4-manifold
Genus
Non-orientable
Spin

ABSTRACT

For a closed 4-manifold X and a knot K in the boundary of punctured X, we define $\gamma_X^0(K)$ to be the smallest first Betti number of non-orientable and null-homologous surfaces in punctured X with boundary K. Note that $\gamma_{S^4}^0$ is equal to the non-orientable 4-ball genus and hence γ_X^0 is a generalization of the non-orientable 4-ball genus. While it is very likely that for given X, γ_X^0 has no upper bound, it is difficult to show it. In fact, even in the case of $\gamma_{S^4}^0$, its non-boundedness was shown for the first time by Batson in 2012. In this paper, we prove that for any Spin 4-manifold X, γ_X^0 has no upper bound.

1. Introduction

Throughout this paper, we work in the smooth category, all 4-manifolds are orientable, oriented and simply-connected, and all surfaces are compact. If X is a closed 4-manifold, punc$X$ denotes X with an open 4-ball deleted.

In [7], we defined the non-orientable genera of knots in punc$X$ as follows.

Definition. Let X be a closed 4-manifold and $K \subset \partial(\text{punc}X)$ ($\cong S^3$) a knot. *The non-orientable X-genus $\gamma_X(K)$ of K* is the smallest first Betti number of any non-orientable surface $F \subset \text{punc}X$ with boundary K. Moreover, we define $\gamma_X^0(K)$ to be the smallest first Betti number of any non-orientable surface $F \subset \text{punc}X$ with boundary K which represents zero in $H_2(\text{punc}X, \partial(\text{punc}X); \mathbb{Z}_2)$.

We note that both $\gamma_X(K)$ and $\gamma_X^0(K)$ are generalizations of the non-orientable 4-ball genus $\gamma_4(K)$, which is the smallest first Betti number of any non-orientable surface in B^4 with boundary K. In this paper, we investigate the following problem.

Problem 1. For a given 4-manifold X, do γ_X and γ_X^0 have upper bounds?

E-mail address: m131708n@st.u-gakugei.ac.jp.

http://dx.doi.org/10.1016/j.topol.2015.02.008
0166-8641/

While γ_4 has been investigated since 1975 [9], it is difficult to evaluate γ_4 and Problem 1 had remained open even in the case of γ_4 until recently. The best reference for related studies is [2]. In 2012, Batson [1] gave a negative answer of Problem 1 in the case of γ_4 by using Heegaard Floer homology theory. (In fact, Batson proved that for any positive integer k, there exists a knot K which satisfies $\gamma_4(K) = k$.) In 2014, Tange and the author [7] applied Batson's idea to the case of $\gamma^0_{n\mathbb{C}P^2}$ and proved that $\gamma^0_{n\mathbb{C}P^2}$ also has no upper bound. On the other hand, Suzuki [8] and Norman [6] proved that if X is diffeomorphic to $S^2 \times S^2$ or $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, then any knot bounds a disk in punc$X$. This result implies that for many 4-manifolds such as $(S^2 \times S^2)\#Y$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}\#Y$, γ_X is bounded above by 1, where Y denotes any 4-manifold. Then, it seems natural to consider for a manifold X with γ_X bounded, whether or not γ^0_X also has upper bound. In this paper, we prove the following theorem that gives a negative answer of this question.

Theorem 1.1. *If X is a Spin 4-manifold, then γ^0_X has no upper bound.*

By this theorem, it follows that for any Spin 4-manifold X, $\gamma_{(S^2\times S^2)\#X}$ is bounded above by 1 and $\gamma^0_{(S^2\times S^2)\#X}$ has no upper bound. Moreover, this theorem gives an alternate proof of non-boundedness of γ_4 which is obtained from Furuta's 10/8-theorem [3] instead of Heegaard Floer homology theory.

2. Construction of a lower bound for γ^0_X

Let Y be a closed 4-manifold. For any knot $K \subset \partial(\text{punc}Y)$, we define $\text{Char}(Y, K)$ a set of characteristic homology classes in $H_2(\text{punc}Y, \partial(\text{punc}Y); \mathbb{Z})$ which are represented by disks in punc$Y$ with boundary K. In this section, we construct a new lower bound of γ^0_X for any Spin 4-manifold X which consists of the knot signature, an element of $\text{Char}(Y, K)$ and invariants for X.

Proposition 2.1. *Let X be a closed Spin 4-manifold and K a knot in $\partial(\text{punc}X)$. Then for any closed 4-manifold Y and any element η of $\text{Char}(Y, K)$, we have*

$$3\gamma^0_X(K) \geq \left|\sigma(K) - \frac{\eta\cdot\eta}{2} + \frac{\sigma(X)+\sigma(Y)}{2}\right| - 4\max(\beta_2^+(X)+\beta_2^-(Y), \beta_2^-(X)+\beta_2^+(Y)) - \beta_2(X),$$

where $\sigma(K)$ and $\sigma(X)$ is the signature of K and X respectively, $\eta\cdot\eta$ is the self-intersection number of η, and β_2^+ (resp. β_2^-) is the rank of positive (resp. negative) part of the intersection form of X.

In order to prove Proposition 2.1, we need the following lemma.

Lemma 2.2. *Let X be a closed 4-manifold, K a knot in the boundary of* punc$X$*, and N a non-orientable surface in* punc$X$ *with $\partial N = K$ which represents a characteristic homology class in $H_2(\text{punc}X, \partial(\text{punc}X); \mathbb{Z}_2)$. Then for any closed 4-manifold Y and any element η of $\text{Char}(Y, K)$, we have*

$$\beta_1(N) \geq \frac{|e(N) - \eta\cdot\eta - \sigma(X) + \sigma(Y)|}{4} - 2\max(\beta_2^+(X)+\beta_2^-(Y), \beta_2^-(X)+\beta_2^+(Y)).$$

Proof. We first prove the lemma in the case when $\beta_1(N)$ is odd. In this case, by applying the arguments of [10, Connecting Lemma II] and [1, Proposition 3], we have a new orientable surface F in $\text{punc}(X\#(S^2\times S^2))$ with boundary K which satisfies the following:

(1) $\beta_1(F) = \beta_1(N) - 1$,
(2) $e(F) = e(N) + 2\varepsilon$ for some $\varepsilon = \pm 1$, and
(3) F represents a characteristic homology class.