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Iterates of dynamical systems on compact metrizable countable spaces ☆

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A B S T R A C T

Given a dynamical system (X, f), we let $E(X, f)$ denote its Ellis semigroup and $E(X, f)^* = E(X, f) \setminus \{f^n : n \in \mathbb{N}\}$. We analyze the Ellis semigroup of a dynamical system having a compact metric countable space as a phase space. We show that if (X, f) is a dynamical system such that X is a compact metric countable space and every accumulation point of X is periodic, then either all functions of $E(X, f)^*$ are continuous or all functions of $E(X, f)^*$ are discontinuous. We describe an example of a dynamical system (X, f) where X is a compact metric countable space, the orbit of each accumulation point is finite and $E(X, f)^*$ contains both continuous and discontinuous functions.

1. Introduction

We start the paper by fixing some standard notions and terminology. Let (X, f) be a dynamical system. The *orbit* of x, denoted by $\mathcal{O}_f(x)$, is the set $\{f^n(x) : n \in \mathbb{N}\}$, where f^n is f composed with itself n times. A point $x \in X$ is called a *periodic point* of f if there exists $n \geq 1$ such that $f^n(x) = x$, and x is called *eventually periodic* if its orbit is finite. The *ω-limit set* of $x \in X$, denoted by $\omega_f(x)$, is the set of points

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$y \in X$ for which there exists an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $f^{n_k}(x) \to y$. For each $y \in \mathcal{O}_f(x)$, $\omega_f(y) = \omega_f(x)$. If $\mathcal{O}_f(y)$ contains a periodic point x, then $\omega_f(y) = \mathcal{O}_f(x)$. We denote by $\mathcal{N}(x)$ the collection of all the neighborhoods of x, for each $x \in X$. The set of all accumulation points of X, the derivative of X, is denoted by X'. We remark that the countable ordinal space $\omega^2 + 1$ is homeomorphic to the compact metric subspace $Y = \{1 - \frac{1}{n} : n \in \mathbb{N} \setminus \{0\}\} \cup \{1\} \cup (\bigcup_{n\in\mathbb{N}} A_n)$ of $\mathbb{R}$, where A_n is an increasing sequence contained in $(1 - \frac{1}{n-1}, 1 - \frac{1}{n})$ such that $A_n \longrightarrow 1 - \frac{1}{n}$, for each $n \in \mathbb{N}$ bigger than 1. The Stone–Čech compactification $\beta(\mathbb{N})$ of $\mathbb{N}$ with the discrete topology will be identified with the set of ultrafilters over $\mathbb{N}$. Its remainder $\mathbb{N}^* = \beta(\mathbb{N}) \setminus \mathbb{N}$ is the set of all free ultrafilters on $\mathbb{N}$, where, as usual, each natural number n is identified with the fixed ultrafilter consisting of all subsets of $\mathbb{N}$ containing n. For $A \subseteq \mathbb{N}$, A^* denotes the collection of all $p \in \mathbb{N}^*$ such that $A \in p$.

In our dynamical systems (X, f) the space X will be compact metric and $f : X \to X$ will be a continuous map. A very useful object to study the topological behavior of the dynamical system (X, f) is the so-called *Ellis semigroup* or *enveloping semigroup*, introduced by Ellis [4], which is defined as the pointwise closure of $\{f^n : n \in \mathbb{N}\}$ in the compact space X^X with composition of functions as its algebraic operation. The Ellis semigroup, denoted by $E(X, f)$, is equipped with the topology inhered from the product space X^X. Enveloping semigroups have played a very crucial role in topological dynamics and they are from an active area of research (see, for instance, the survey article [8]).

The motivation of our work is the fact that for some spaces either all functions of $E(X, f)^*$ are continuous or all are discontinuous. Namely, this holds when X is a convergent sequence with its limit point [7] (see also [6]) and for $X = [0, 1]$ as it was recently shown by P. Szuca [10]. In this direction, we will show that it also happens for any dynamical system (X, f) where X is a compact metrizable countable space such that every accumulation point of X is periodic. We also present an example of a dynamical system (X, f) where X is the ordinal space $\omega^2 + 1$ such that $E(X, f) \setminus \{f^n : n \in \mathbb{N}\}$ contains continuous and also discontinuous functions and each accumulation point of X is eventually periodic. This answers a question posed in [7].

Now we recall a convenient description of $E(X, f)$ in terms of the notion of p-limits where p is an ultrafilter on the natural number $\mathbb{N}$. Given $p \in \mathbb{N}^*$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in a space X, we say that a point $x \in X$ is the *p-limit point* of the sequence, in symbols $x = p - \lim_{n\to\infty} x_n$, if for every neighborhood V of x, $\{n \in \mathbb{N} : f^n(x) \in V\} \in p$. Observe that a point $x \in X$ is an accumulation point of a countable set $\{x_n : n \in \mathbb{N}\}$ of X iff there is $p \in \mathbb{N}^*$ such that $x = p - \lim_{n\to\infty} x_n$. It is not hard to prove that each sequence of a compact space always has a p-limit point for every $p \in \mathbb{N}^*$. The notion of a p-limit point has been used in topology and analysis (see for instance [2] and [5, p. 179]).

A. Blass [1] and N. Hindman [9] formally established the connection between "the iteration in topological dynamics" and "the convergence with respect to an ultrafilter" by considering a more general iteration of the function f as follows: Let X be a compact space and $f : X \to X$ a continuous function. For $p \in \mathbb{N}^*$, the p-iterate of f is the function $f^p : X \to X$ defined by

$$f^p(x) = p - \lim_{n\to\infty} f^n(x),$$

for all $x \in X$. The description of the Ellis semigroup in terms of the p-iterates is then the following:

$$E(X, f) = \{f^p : p \in \beta\mathbb{N}\}$$
$$f^p \circ f^q = f^{q+p} \quad \text{for each } p, q \in \beta\mathbb{N} \text{ (see [1,9])}.$$

The paper is organized as follows. The second section is devoted to prove some basic results that will be used in the rest of the paper. In the third section, we show our main results about $E(X, f)$ when X is a compact metric countable space and each element of X' is a periodic point of f. In the forth section, we construct a dynamical system (X, f) in which all accumulation points are eventually periodic