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Local connectedness in topological groups

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A B S T R A C T

This paper gives several conditions sufficient for a topological group to be locally connected, locally path connected, or almost locally path connected. Structural theorems for non-locally connected groups arise, such as a limitation on the types of subspaces that can occur as somewhere dense subsets, and the meagerness of conjugacy classes in many instances. Improvements are given to certain open, or almost open, mapping theorems. New observations are made about the associated locally path connected topology of a topological group.

In the solution of Hilbert's fifth problem (see [6,11,19,20]) conditions were given for locally compact groups to be locally path connected, and indeed locally Euclidean and Lie. In the ensuing years, with much of the theory of locally compact groups having been completed, attention has turned to more general topological groups. Though much has been done, understanding the structure, of say Polish groups, or topological groups in general, is far from complete.

In [17] it was shown that all path-connected Polish topological groups are locally path connected. In [18] it was shown that every topological group with a second countable, second category path component is

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locally connected. The purpose of the present article is to discover the most basic principals underlying these results. Consequently, these results are strengthened, and the entire theory of local connectedness, or local path connectedness, of topological groups is given a more unified perspective. Numerous new conditions are given for a group to be locally connected. In the section on path connectedness, in addition to new results, connections with the classical results are given.

Some structural theorems for non-locally connected groups naturally appear, such as a limitation on the types of subspaces that can be somewhere densely imbedded, and the meagerness of conjugacy classes in many cases.

Improvements are given to certain commonly encountered open, or almost open, mapping theorems. Additionally, new observations are made about the *associated locally path connected topology* of a topological group.

The paper concludes with open questions whose answers would add significantly to the theory.

1. Preliminaries

Throughout, G will be a connected, T_0 (and hence Hausdorff and regular) topological group, $\mathcal{U}_e$ the set of all open sets containing e (the identity element), D the diagonal in $G \times G$, and $\mathfrak{L}$ the left uniformity on G, with base:

$$\{U_L \mid U \in \mathcal{U}_e\} \quad \text{where } U_L = \{(x, y) \mid x^{-1}y \in U\}$$

Let F be the function from $G \times G$ onto G given by:

$$F(x, y) = x^{-1}y$$

Notice that F is continuous, open, and carries the diagonal D of $G \times G$ to e. Properties of $G \times G$ near D immediately translate to properties of G near e, and it is helpful to keep this imagery in mind. All of the results of the paper which rely on $\mathfrak{L}$ and F have corresponding results, which we have chosen to omit, for the right uniformity $\mathfrak{R}$ and the corresponding map $F_R(x, y) = yx^{-1}$.

We will say that a subset C of a topological space X is *somewhere dense* in X if $(\overline{C})^\circ \neq \emptyset$. If $V \subseteq \overline{C}$ is an open set, we will say that C *is dense in* V. If C is dense in V, it is dense in every open subset of V. Here are a couple of useful lemmas, included for want of a reference.

Lemma 1. *If C is somewhere dense in a topological space X, and $\mathcal{U}$ is a cover of C by subsets that are relatively open in C, then for some $U \in \mathcal{U}$, $(\overline{U})^\circ \neq \emptyset$.*

Proof. By hypothesis, C is dense in some nonempty open set V. If we take any $c \in C \cap V$, then c lies in some $U \in \mathcal{U}$, and $U = W \cap C$ for some open set W. Therefore $c \in W \cap V$. Since C is dense in V, it is also dense in $W \cap V$. Therefore,

$$W \cap V \subseteq \overline{C \cap W \cap V} \subseteq \overline{U}. \qquad \square$$

Lemma 2. *If $B \subseteq X$ is somewhere dense in X, and $\{A_\alpha\}$ is a collection of subsets of B such that in the relative topology of B, the interiors of the closures of the A_α cover B, then for some α, A_α is somewhere dense in X.*

Proof. By hypothesis, the sets $\mathrm{Int}_B(\mathrm{Cl}_B(A_\alpha))$ cover B. Thus by the previous lemma, there is an α such that $\mathrm{Int}_B(\mathrm{Cl}_B(A_\alpha))$ is somewhere dense in X. Thus, there is a nonempty open set V such that