Topology and its Applications 180 (2015) 124–131

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Condensations of paratopological groups

Iván Sánchez

Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Del. Iztapalapa, C.P. 09340, Mexico, D.F., Mexico

ARTICLE INFO

Article history:
Received 25 June 2014
Received in revised form 26 October 2014
Accepted 15 November 2014
Available online 27 November 2014

MSC:
22A05
54A25
54D20
54G20
54H11

Keywords:
Semitopological group
Paratopological group
Condensation
ω-narrow
Hausdorff number

ABSTRACT

We prove that every Hausdorff Lindelöf paratopological group with countable pseudocharacter admits a condensation onto a separable metrizable space. This result resolves a problem of M. Tkachenko. Also, we show that every regular (Hausdorff) ω-narrow semitopological (paratopological) group with countable Hausdorff number and countable pseudocharacter condenses onto a second countable Urysohn space.

We show that each regular precompact paratopological group of countable pseudocharacter admits a continuous isomorphism onto a metrizable separable topological group. Also, we construct a Hausdorff precompact paratopological group with countable pseudocharacter which cannot be condensed onto a Hausdorff first countable space.

1. Introduction

A *paratopological* (*semitopological*) group is a group endowed with a topology for which multiplication in the group is jointly (separately) continuous. If, additionally, the inversion in a paratopological group is continuous, then it is called a *topological group*.

In 1953, Katz showed that a topological group is topologically isomorphic to a subgroup of a topological product of first-countable (metrizable) topological groups if and only if it is ω-balanced (see [11]). This fact implies that each ω-balanced topological group with countable pseudocharacter admits a continuous isomorphism onto a metrizable topological group. Another important result about condensations in topological groups was proved by Arhangel'skii in 1980: every Hausdorff topological group of countable pseudocharacter is submetrizable, i.e., admits a weaker metrizable (not necessarily topological group) topology (see [1]).

E-mail address: isr.uami@gmail.com.

http://dx.doi.org/10.1016/j.topol.2014.11.009

According to Guran's theorem in [9], a topological group is topologically isomorphic to a subgroup of a topological product of second-countable topological groups if and only if it is ω-narrow. It follows that every ω-narrow topological group with countable pseudocharacter admits a continuous isomorphism onto a second countable topological group. However, we cannot extend results mentioned previously to the class of paratopological groups: we construct an Abelian Hausdorff precompact paratopological group which cannot be condensed onto a first-countable space (see Example 2.15).

Some results about condensations of paratopological group appear in [2,3,14,16,18–20,22].

In this paper, we study condensations from a Hausdorff paratopological group onto an Urysohn second countable space or a separable metrizable space.

We denote by $l(X)$ the Lindelöf number and by $\psi(X)$ the pseudocharacter of a space X. In [22], M. Tkachenko posed the question:

A) Does a Hausdorff (or regular) paratopological group G with $l(G)\psi(G) \leq \omega$ admit a continuous bijection onto a Hausdorff space with a countable base?

In Theorem 2.7, we solve Problem A). The problem was also solved independently by P.Y. Li, L.-H. Xie, and S. Lin (see [12]). Their proof used a theorem from general topology: Every Hausdorff paracompact space with a G_δ-diagonal is submetrizable. Our proof, from our dissertation [15], is different in that it is a direct proof that does not use that theorem from general topology.

We show that every regular (Hausdorff) ω-narrow semitopological (paratopological) group with countable Hausdorff number and countable pseudocharacter condenses onto a second countable Urysohn space (see Theorem 2.17 and Corollary 2.19).

We prove that each regular precompact paratopological group of countable pseudocharacter admits a continuous isomorphism onto metrizable separable topological group (see Corollary 2.14). Also, we construct a Hausdorff precompact paratopological group with countable pseudocharacter which cannot be condensed onto a Hausdorff first countable space (see Example 2.15).

2. Condensations

The following two definitions play an important role in this paper.

Definition 2.1. ([5]) A semitopological group G is *ω-narrow* if for every neighborhood U of the identity e in G, there exists a countable subset $A \subseteq G$ such that $AU = UA = G$.

Definition 2.2. ([20]) For a Hausdorff semitopological group G with identity e, the *Hausdorff number* of G, denoted by $Hs(G)$, is the minimum cardinal number κ such that for every neighborhood U of e in G, there exists a family γ of neighborhoods of e such that $\bigcap_{V\in\gamma} VV^{-1} \subseteq U$ and $|\gamma| \leq \kappa$.

According to [2], every Tychonoff ω-narrow semitopological group of countable π-character is submetrizable. Here, we present a similar result.

Theorem 2.3. *Let G be a completely regular ω-narrow semitopological group such that $Hs(G)\psi(G) \leq \omega$. Then G can be condensed onto a separable metrizable space.*

Proof. By hypothesis, there exists a family $\{W_n : n \in \omega\} \subseteq \mathcal{N}(e)$ which satisfies $\bigcap_{n\in\omega} W_n W_n^{-1} = \{e\}$. Since G is completely regular, for each $n \in \omega$, we can find a continuous function $f_n: G \to [0,1]$ such that $f_n(e) = 1$ and $f_n(G \setminus W_n) \subseteq \{0\}$. Put $V_n = \{x \in G : f_n(x) \neq 0\}$. Clearly V_n is an open neighborhood of the identity contained in W_n for every $n \in \omega$, so $\bigcap_{n\in\omega} V_n V_n^{-1} = \{e\}$. Since G is ω-narrow, for each