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The extensions of some convergence phenomena in topological groups ☆

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A B S T R A C T

In this paper sequentially compact sets, weakly first-countable sets and generalized metric sets in extensions of topological groups are studied. Some three space properties on convergence phenomena are obtained. It is shown that (1) if H is a closed subgroup of a topological group G such that all sequentially compact subsets of both the group H and the quotient space G/H are sequential, then all sequentially compact subsets of G are sequential; (2) let H be a closed and second-countable subgroup of a topological group G, then G is a topological sum of $\aleph_0$-subspaces if the quotient space G/H is a local $\aleph_0$-space; (3) let H be a locally compact and metrizable subgroup of a topological group G, then G is sequential if the quotient space G/H is sequential.

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1. Introduction

One of the main operations on topological groups is that of taking quotient groups. Many non-trivial examples and counterexamples arise as quotients of relatively simple and well-known topological groups. This operation has been the subject of an intensive and thorough study; but there still exists a wealth of interesting open problems related to the behavior of different topological and algebraic properties under taking quotients [2].

The following general question is considered in [1]. Let H be a closed subgroup of a topological group G, and G/H the quotient space. Suppose that both H and G/H belong to some nice class of topological spaces. When can we conclude that G is in the same class? The group G is called an *extension of the group H by the quotient space G/H* [27].

In 1949, J.P. Serre [21] proved that if H is a closed subgroup of a topological group G, and both the spaces H and G/H are locally compact, then the topological group G is locally compact. This classical result on the extension of properties from G/H to G induces a study of the above general question. Suppose that H is a closed subgroup of a topological group G, and G/H is the corresponding quotient space. A (topological, algebraic, or a mixed nature) property $\mathcal{P}$ is said to be a *three space property* [5] if, for every topological group G and a closed subgroup H of G, the fact that both spaces H and G/H have $\mathcal{P}$ implies that G also has $\mathcal{P}$. The Serre's theorem implies that local compactness is a three space property. In fact, compactness, completeness, pseudocompactness, connectedness and metrizability are three space properties in the class of topological groups, but countable compactness is not [4].

Recently, A.V. Arhangel'skiĭ, M. Bruguera, M.G. Tkachenko and V.V. Uspenskij [2–5,25] discovered a series of results on the extensions of topological groups with respect to closed invariant subgroups, locally compact subgroups or locally compact metrizable subgroups. These results show that finding three space properties is one of interesting questions in topological groups. Some problems are still open in this direction.

Question 1.1 *([2, Open problem 1.5.1]). Characterize (or find the typical properties) of compact spaces that can be represented as quotients of topological groups with respect to closed metrizable subgroups.*

Question 1.2 *([2, Open problem 9.10.1]). Suppose that H is a closed invariant subgroup of a topological group G, and all compact subsets of the groups H and G/H are sequentially compact. Does G have the same property?*

Question 1.3 *([2, Open problem 9.10.3]). Let all compact subsets of the groups H and G/H be Fréchet–Urysohn. Does the same hold for compact subsets of G?*

Convergence is a basic research object in general topology. It is natural and quite plausible to expect that certain convergence properties of topological spaces should become stronger in topological groups [22]. The most obvious example of this phenomena is that first-countability becomes equivalent to metrizability in topological groups [2]. Some convergence properties in topological groups were introduced in [2]. In this paper we consider the extensions of some convergence properties in topological groups. In Section 2 the three space question for sequentially compact sets is discussed related to Question 1.2. It is shown that if H is a closed subgroup of a topological group G such that all compact subsets of the group H are first-countable, then all compact subsets of G are Fréchet if so is G/H, which gives a partial answer to Question 1.3. In Section 3 the quotients with respect to second-countable subgroups of topological groups are considered. It is proved that if H is a closed second-countable subgroup of a topological group G, then G is a topological sum of $\aleph_0$-subspaces if G/H is a local $\aleph_0$-space. In Section 4 the quotients with respect to locally compact subgroups of topological groups are studied. It is shown that a topological group G is a sequential space if H a locally compact metrizable subgroup of G and the quotient space G/H is sequential.