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Common fixed points for commuting mappings in hyperconvex spaces

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ABSTRACT

In a bounded hyperconvex metric space M we prove common fixed point results for nonexpansive mappings $f : M \to M$ and $F : M \to 2^M$ such that the mappings either commute or commute weakly. Our results provide hyperconvex space analogues of similar common fixed point theorems in Banach and CAT(0) spaces. Our method for weakly commuting mappings uses the hyperconvexity of $\mathcal{N}(M)$ the space of nonexpansive mappings of M with the sup metric. We show that if $F : M \to 2^M$ has externally hyperconvex values then the set of all nonexpansive selections of F is an externally hyperconvex subset of $\mathcal{N}(M)$.

1. Introduction

In a hyperconvex metric space (M, d) we consider mappings $f : M \to M$ and $F : M \to 2^M$ that are said to *commute* if $f(F(x)) \subset F(f(x))$, and to *commute weakly* if $f(\partial_M F(x)) \subset F(f(x))$, for $x \in M$. When f and F are nonexpansive and satisfy one of these conditions we address the following problems:

(1) *common fixed points*
under what conditions does there exist a point $z \in M$ such that $z = f(z) \in F(z)$? and

(2) *invariant approximation*
for a set $C \subset M$ under what conditions does there exist a point $z \in C$ such that if p is a common fixed point of f and F then $d(p, z) = d(p, C)$ and $z = f(z) \in F(z)$?

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For results on these problems for commuting or weakly commuting nonexpansive mappings in Banach spaces see for example [4,7]. Similar results for these problems in CAT(0) spaces are found in [5,11,12]. In both Banach and CAT(0) spaces a common method of proof depends in part on the closure and convexity of $\mathrm{Fix}(f)$ and the fact that under certain assumptions $F(x) \cap \mathrm{Fix}(f)$ is a nonempty closed convex subset for $x \in M$. However, in hyperconvex metric spaces $\mathrm{Fix}(f)$ is only known to be hyperconvex and the latter approach is not possible. Thus, in considering Problem 1 for weakly commuting mappings we use a different method based on the hyperconvexity of the family $\mathcal{N}(M)$ of all bounded nonexpansive mappings of M into itself with the supremum metric. This approach was initiated in [12] to obtain a common fixed point for commuting mappings in hyperconvex spaces and is extended here to include weakly commuting mappings. We show that if $F : M \to 2^M$ is nonexpansive with externally hyperconvex values then the set of all nonexpansive selections of F is an externally hyperconvex subset of $\mathcal{N}(M)$.

2. Preliminaries

A metric space (M, d) is said to be *hyperconvex* if for any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of positive real numbers satisfying $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$, it is the case that $\bigcap_\alpha B(x_\alpha; r_\alpha) \ne \emptyset$, where $B(x; r)$ denotes a closed ball with center $x \in M$ and radius r. Hyperconvex metric spaces were introduced in [1]. For an understanding of the geometry, methods of nonlinear analysis, and especially metric fixed point theory of these spaces, see for example [2,3,6,8,9,13,14].

An *admissible* subset of a hyperconvex metric space M is a set of the form $\bigcap_\alpha B(x_\alpha; r_\alpha)$. For any subset S of M we define $\mathrm{dist}(x, S) = \inf_{y \in S} d(x, y)$. A subset S of a metric space M is said to be *externally hyperconvex* if given any family $\{x_\alpha\}$ of points in X and any family $\{r_\alpha\}$ of positive real numbers satisfying $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$ and $\mathrm{dist}(x_\alpha, S) \le r_\alpha$ it follows that $\bigcap_\alpha B(x_\alpha; r_\alpha) \cap S \ne \emptyset$. A subset S of M is *proximinal* if for any $x \in M$ there is a $y \in S$ such that $d(x, y) = \mathrm{dist}(x, S)$. The externally hyperconvex subsets of M are proximinal [1]. We denote by d_H the Hausdorff metric defined for any pair of closed bounded subsets A, B of M by $d_H(A, B) = \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}$, where for a subset S of a metric space, $N_\varepsilon(S) = \{x \in M : d(x, S) \le \varepsilon\}$. A mapping $F : M \to 2^M$ is *nonexpansive* provided $d_H(F(x), F(y)) \le d(x, y)$ for $x, y \in M$, and the fixed point set of F is denoted by $\mathrm{Fix}(F)$. The family $\mathcal{N}(M)$ of all bounded nonexpansive self-mappings of M with the supremum metric $d(f, g) = \sup\{d(f(x), g(x)) : x \in M\}$ is hyperconvex [3, Thm. 3].

3. Commuting mappings

Theorem 1. *Let M be a bounded hyperconvex metric space, and $F : M \to 2^M$ a nonexpansive mapping with nonempty externally hyperconvex values. Let $\{g_\alpha\}$ be a family of commuting nonexpansive self-mappings of M. If each g_α commutes with F, then the family $\{g_\alpha\}$ and F have a common fixed point, and the set of common fixed points is a hyperconvex subset of M.*

Proof. By [8, Cor. 3] the mapping F has a nonempty fixed point set that is hyperconvex. For any $p \in \mathrm{Fix}(F)$, $p \in F(p)$, and since $\{g_\alpha\}$ commutes with F, $g_\alpha(p) \in F(g_\alpha(p))$. This implies that $\mathrm{Fix}(F)$ is invariant under the family $\{g_\alpha\}$. Since the $\{g_\alpha\}$ commute on the hyperconvex set $\mathrm{Fix}(F)$, they have a common fixed point $z = g_\alpha(z) \in F(z)$ [3]. Considering the family $\{g_\alpha\}$ as self-mappings of the hyperconvex set $\mathrm{Fix}(F)$, by [3] the set of common fixed points of the family $\{g_\alpha\}$ is hyperconvex.

The following invariant approximation result is a hyperconvex space version of a similar result for commuting nonexpansive mappings in R-trees [11, Thm. 5.3].

Theorem 2. *Let M be a bounded hyperconvex metric space, $F : M \to 2^M$ a nonexpansive mapping with nonempty externally hyperconvex values, and $\{g_\alpha\}$ a family of commuting nonexpansive self-mappings of M*