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Non-universality of automorphism groups of uncountable ultrahomogeneous structures

Michal Doucha [1]

Institute of Mathematics, Polish Academy of Sciences 00-656 Warszawa, Poland

A R T I C L E I N F O

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A B S T R A C T

In [13], Mbombo and Pestov prove that the group of isometries of the generalized Urysohn space of density κ, for uncountable κ such that $\kappa^{<\kappa} = \kappa$, is not a universal topological group of weight κ. We investigate automorphism groups of other uncountable ultrahomogeneous structures and prove that they are rarely universal topological groups for the corresponding classes. Our list of uncountable ultrahomogeneous structures includes random uncountable graph, tournament, linear order, partial order, group. That is in contrast with similar results obtained for automorphism groups of countable (separable) ultrahomogeneous structures. We also provide a more direct and shorter proof of the Mbombo–Pestov result.

0. Introduction

In [14], Uspenskij proved that the group of all isometries of the Urysohn universal metric space with pointwise convergence topology is a universal topological group of countable weight. The proof consists of two steps. First, realizing that any topological group (of countable weight) is isomorphic to a subgroup of the group of isometries of some (separable) metric space. Second and most important, that every group of isometries of some separable metric space with pointwise convergence topology embeds into the group of isometries of the Urysohn space.

The latter fact was then observed in place of many other ultrahomogeneous (discrete) structures (we refer the reader to the next section for definitions). Let M be a (discrete or metric) ultrahomogeneous structure and let $\mathrm{Aut}(M)$ denote the group of all automorphisms of M equipped with the pointwise convergence

E-mail address: m.doucha@post.cz.
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topology. Is then $\mathrm{Aut}(M)$ the universal group for the class of groups that can be represented as automorphism groups of some substructure of M? More precisely, let G be a topological group that is isomorphic to (a subgroup of) the group of automorphisms of some substructure $A \subseteq M$. Does G homeomorphically embed into $\mathrm{Aut}(M)$? E. Jaligot was probably the first who, in [9], formulated this problem in general and proved it for M being the random tournament. The same was proved by Bilge and Jaligot in [3] when M is a random K_n-free graph, for $n \geq 3$. Bilge and Melleray in [4] generalized this for ultrahomogeneous structures when the corresponding Fraïssé class has the free amalgamation property. Kubiś and Mašulović in [11] gave another examples and formulated the problem in the language of category theory. Another metric versions are the following: Gao and Shao in [7] proved it for a universal and ultrahomogeneous separable R-ultrametric space, where $R \subseteq \mathbb{R}^+$ is some countable set of distances. And recently, Ben Yaacov in [2] when proving that the group of linear isometries of the Gurarij space is a universal Polish group proved that this linear isometry group is also universal for the corresponding class of groups (the universality as a Polish group then follows from the fact that every Polish group is isomorphic to a subgroup of linear isometries of some separable Banach space).

It is an open question whether there is actually a counterexample, i.e. a countable ultrahomogeneous structure M and a substructure A such that $\mathrm{Aut}(A)$ does not topologically embed into $\mathrm{Aut}(M)$.

For certain uncountable cardinals, namely those uncountable κ such that $\kappa^{<\kappa} = \kappa$, one can have the (discrete and metric) Fraïssé theory as well. It means one can produce structures of cardinality κ that are κ-ultrahomogeneous, i.e. any partial isomorphism between two substructures of cardinality strictly less than κ extends to the full automorphism. Katětov, in [10], proved the existence of the generalized Urysohn space $\mathbb{U}_\kappa$, for κ as above, of weight κ.

Mbombo and Pestov in [13], when checking whether $\mathrm{Iso}(\mathbb{U}_\kappa)$ might be the universal topological group of weight κ (the existence of such groups of any uncountable weight is still not known), found out that situation changes at the higher cardinality. They proved that $\mathrm{Iso}(\mathbb{U}_\kappa)$ is not the universal topological group of weight κ, i.e. equivalently, there are metric spaces X of density at most κ such that $\mathrm{Iso}(X)$ does not topologically embed into $\mathrm{Iso}(\mathbb{U}_\kappa)$.

In this paper, we focus on the general universality problem in the "uncountable Fraïssé theoretic setting", as considered by Mbombo and Pestov in the particular case of the uncountable Urysohn space. We show that while in the countable case the norm is that the automorphism group of an ultrahomogeneous structure is universal for the corresponding class of groups, in the uncountable case the norm seems to be the opposite – it is probably very rarely universal.

Let us list here some particular structures for which we can prove it.

Theorem 0.1. *Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Let M be one of the following structures: random graph of cardinality κ, random partial order of cardinality κ, random linear order of cardinality κ, random tournament of cardinality κ and random group of cardinality κ. Then $\mathrm{Aut}(M)$ is not universal. There is a substructure $A \subseteq M$ such that $\mathrm{Aut}(A)$ does not continuously embed into $\mathrm{Aut}(M)$.*

We also provide a shorter and more direct proof of the Mbombo–Pestov result.

1. Preliminaries

Let us explain more precisely the problem from the introduction here. The set-up is the following: let L be some fixed countable language and let M be a first-order L-structure, either countable discrete or separable metric (as is the case with the Urysohn space), that is ultrahomogeneous. These are precisely (metric) Fraïssé limits of (metric) amalgamation classes. We refer the reader to [8] for information about Fraïssé theory, to [1] for information about metric Fraïssé theory and to [12] for a survey on ultrahomogeneous structures.