Topology and its Applications 178 (2014) 125–135

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On transitive expansive homeomorphisms of the plane

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ARTICLE INFO

Article history:
Received 7 January 2014
Received in revised form 6 July 2014
Accepted 2 September 2014
Available online 20 September 2014

Keywords:
Transitive homeomorphisms
Expansiveness
Uniform expansiveness
Lyapunov functions
Fathi's metric

ABSTRACT

We study the existence of transitive expansive homeomorphisms of $\mathbb{R}^2$. We prove that there are no (orientation preserving) transitive and uniformly expansive homeomorphisms of the plane.

1. Introduction

Expansiveness and transitivity are well known sources of chaotic behavior. Indeed any of these properties alone generates a complicated dynamical behavior when the ambient space is a compact manifold, see for instance [11,12,5,4,9,16]. For the non-compact case this is not necessarily true. For instance, a homothety H of center the origin O and ratio $r > 1$ defined on $\mathbb{R}^2$ is expansive but its dynamics is trivial, all points except the origin diverge to ∞ by forward iteration by H and converge to O by backward iteration. For the case of transitiveness the first thing is that it is not evident if it is possible to exhibit a transitive homeomorphism defined on the plane. The first to construct such examples seems to have been L.G. Shnirelman and later A.S. Besicovitch. In his article [2] Besicovitch built an example of a transitive homeomorphism of the plane. He proved that all forward orbits departing from a straight line are dense in $\mathbb{R}^2$ and conjectured that all points of the plane had this property. Later in [3] Besicovitch exhibited orbits of the same example that are not transitive disproving the conjecture. In fact there is a dense subset of $\mathbb{R}^2$ such that its forward orbits

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http://dx.doi.org/10.1016/j.topol.2014.09.004

are not dense in $\mathbb{R}^2$ (see Lemma 2.6). It is shown in the book by Alpern and Prasad [1] that examples like that of [2] are maximally chaotic in the sense of Alpern and Prasad (see [1, Chapter 17]).

Mendes in [13] studied Anosov diffeomorphisms $f : \mathbb{R}^2 \to \mathbb{R}^2$ and proved that in that case $\Omega(f)$ is empty or just a single fixed point. Therefore such a diffeomorphism is expansive but not transitive.

Groisman has studied in [7,8] expansive homeomorphisms h of the plane giving conditions under which such a homeomorphism is conjugate to a linear hyperbolic transformation and in that case $\Omega(h)$ is a single fixed point or to a translation of the plane. In both cases the non-wandering set is trivial.

The question arises whether it is possible for a homeomorphism to share both, expansiveness and transitivity. But if a homeomorphism has both properties at the same time we should expect rich dynamical properties even in non-compact spaces. In this paper we investigate if there exists a homeomorphism of $\mathbb{R}^2$ with both properties at the same time. The answer is negative if non-trivial compact connected stable and unstable sets can be built in a neighborhood of the fixed point that necessarily exists due to Brouwer theory of homeomorphisms of the plane. This is the case when we ask for $f : \mathbb{R}^2 \to \mathbb{R}^2$ to be *uniformly* expansive, [15], and transitive (see definitions below).

We wish to pose the following questions:

1. Are there transitive expansive homeomorphisms defined on the plane (dropping the hypothesis of uniformity of expansiveness)?
2. What can be said with respect to transitive expansive homeomorphisms defined on $\mathbb{R}^n$ for $n \geq 3$?

2. Lyapunov stable points

Definition 2.1. We say that the point $x \in \mathbb{R}^2$ is f-transitive if

$$\overline{\mathrm{Orb}(x)} = \mathrm{closure}\bigl(\{f^n(x) : n \in \mathbb{Z}\}\bigr) = \mathbb{R}^2.$$

We say that x is positive (negative) f-transitive if the forward (resp.: backward) orbit by f is dense in $\mathbb{R}^2$, i.e., $\overline{\mathrm{Orb}^+(x)} = \mathbb{R}^2$ (resp.: $\overline{\mathrm{Orb}^-(x)} = \mathbb{R}^2$).

Remark 2.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism that preserves orientation. Hence, since f is assumed to be transitive, from Brouwer theory of homeomorphisms preserving orientation of the plane it follows that there exists a fixed point for f. It is immediate that if there is an f-transitive point then there is a dense set of f-transitive points in $\mathbb{R}^2$.

In the sequel we will omit to mention f when x is an f-transitive point. We will just say that it is a transitive point. Also we will say that f is transitive if it has a transitive point. The properties of transitive homeomorphisms $f : X \to X$ with X a compact manifold are well known. Moreover almost the same proofs are valid when X is a second countable complete metric space.

Lemma 2.2. *If x is transitive then either it is positive transitive or it is negative transitive. Moreover, if $\overline{\mathrm{Orb}^-(x)} = \mathbb{R}^2$ then there is a residual subset $\mathcal{R}$ of $\mathbb{R}^2$ of positive transitive points, i.e., if $y \in \mathcal{R}$ then $\overline{\mathrm{Orb}^+(y)} = \mathbb{R}^2$.*

Proof. Let $x \in \mathbb{R}^2$ be such that $\overline{\mathrm{Orb}(x)} = \mathbb{R}^2$. Then there is a sequence $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$ with $|n_j| \to +\infty$ such that $f^{n_j}(x) \to x$. Hence for every $k \in \mathbb{Z}$ it holds that $f^{n_j+k}(x) \to f^k(x)$ when $j \to +\infty$. Either there is an infinite subsequence of $\{n_j\}$ of positive numbers or one of negative ones.

In the first case, assuming that $\{n_j\}$ itself is of positive numbers in order not to complicate notation, we obtain on account of the density of the x orbit, that given $z \in \mathbb{R}^2$ and $\epsilon > 0$ there is $k \in \mathbb{Z}$ such that