Topology and its Applications 178 (2014) 136–145

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Abrams's stable equivalence for graph braid groups

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A R T I C L E   I N F O

Article history:
Received 5 July 2014
Received in revised form 15 September 2014
Accepted 15 September 2014
Available online 25 September 2014

MSC:
20F36
20F65

Keywords:
Graph braid group
Configuration space
Discrete Morse theory

A B S T R A C T

In his PhD thesis [1], Abrams proved that, for a natural number n and a graph G with at least n vertices, the n-strand configuration space of G, denoted $\mathcal{C}^n(G)$, deformation retracts to a compact subspace, the *discretized* n-strand configuration space, provided G satisfies two conditions: each path between distinct *essential vertices* (vertices of degree not equal to 2) is of length at least $n+1$ edges, and each path from a vertex to itself which is not nullhomotopic is of length at least $n+1$ edges. Using Forman's discrete Morse theory for CW-complexes, we show the first condition can be relaxed to require only that each path between distinct essential vertices is of length at least $n-1$.

1. Introduction

The goal of this paper is to establish sufficient conditions such that a *braid group on a graph* may be studied via a certain CW complex associated to the graph. Let X denote a connected topological space. An *n-strand configuration in X* is an n-point subset of X. The *unordered n-strand configuration space* of X is the space of unordered subsets consisting of n distinct elements of X, and is denoted $U\mathcal{C}^n(X)$. (We use the terms *labeled* and *unlabeled* as synonyms for the terms *ordered* and *unordered*, respectively.) For a positive integer n, the classical braid group B_n is the fundamental group $\pi_1(U\mathcal{C}^n(D^2))$, where D^2 is the 2-dimensional topological disk. Thus, from the configuration-space perspective, a braid is simply a loop in the space $U\mathcal{C}^n(D^2)$. Similarly, the *ordered n-strand configuration space* of D^2, denoted $\mathcal{C}^n(D^2)$, is the space of ordered tuples consisting of n distinct elements of X. The classical *n-strand pure braid group*, denoted PB_n, is the fundamental group of the ordered n-strand configuration space of a disk. Note, the quotient map from the configuration space $\mathcal{C}^n(D^2)$ to the unordered configuration space $U\mathcal{C}^n(D^2)$ induces a short exact sequence $1 \to PB_n \to B_n \to \Sigma_n \to 1$, where Σ_n is the symmetric group on n symbols. For an extensive reference on classical braid groups, see [3].

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http://dx.doi.org/10.1016/j.topol.2014.09.009

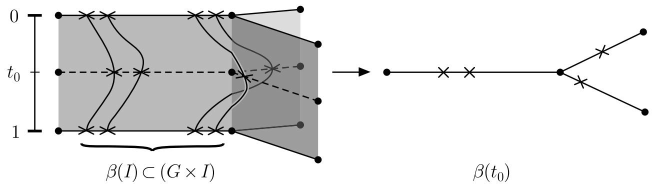

Fig. 1. A nontrivial 4-braid in the cylinder $K_{3,1} \times [0,1]$. At each $t \in [0,1]$, an n-braid defines a configuration of n points of the graph, illustrated by the X's on the graph at right.

In the case of graph braid groups, we let $X = G$ be a connected graph, viewed as a 1-dimensional CW complex. The ordered n-strand configuration space of G is denoted $\mathcal{C}^n(G)$. The *n-strand pure graph braid group* $PB_n(G)$ is the fundamental group $\pi_1(\mathcal{C}^n(G))$. The unordered configuration space is $U\mathcal{C}^n(G)$, and its fundamental group $\pi_1(U\mathcal{C}^n(G))$ is the *n-strand graph braid group* $B_n(G)$. Note that these fundamental groups do not depend on basepoint. Usually the configuration space is connected, but even when it is disconnected the components are all homeomorphic [1].

Graph braid groups, like classical braid groups, can also be viewed as isotopy classes rel endpoints of *braids* (i.e., certain n-tuples of pairwise-disjoint paths) in the cylinder on a topological space. For classical braid groups, this cylinder is $D^2 \times I$, where I is the interval $[0,1]$. In the case of the graph braid group $B_n(G)$, one considers instead braids in the cylinder $G \times I$. Fig. 1 shows a non-trivial braid in $G \times I$, where G is isomorphic to the complete bipartite graph $K_{3,1}$. A braid $\beta : I \to G \times I$ can be thought of as describing the simultaneous and continuous movements of the n strands, or *tokens*, without collisions, on G. To each $t \in I$, the map β associates a configuration $\beta(t)$ of the n strands on G. Since β is a loop in the (ordered/unordered) configuration space, it follows that the configurations $\beta(0)$ and $\beta(1)$ are equal. For example, in Fig. 1, $\beta(0) = \beta(1)$ as unordered configurations.

Besides providing a class of interesting mathematical objects, graph braid groups have real-world applications that have been discussed in [2] and [12]. An example often given is that of a fleet of mobile robots inside a factory, whose movement is confined to a network of track or guide tape. For an idealized robot of infinitesimal size, the configuration space of points on a graph shaped like the track network exactly describes the space of configurations of the fleet of robots.

Abrams introduces the notion of a *discretized* configuration space of n strands on a graph G in his PhD thesis [1]. This is a compact subspace of a configuration space of G, consisting of only those n-strand configurations x in which, for each pair of strands in x, every path in G between the two strands contains at least one full edge of G. Note that $\mathcal{C}^n(G)$ is a subspace of the cubical complex $\prod^n G$, but does not inherit its CW structure, as it is not a compact subspace. In contrast, the *discretized labeled configuration space* of a graph G, denoted $\mathcal{D}^n(G)$, has a CW complex structure as a subcomplex of $\prod^n G$, as does the *discretized unlabeled configuration space*, denoted $U\mathcal{D}^n(G)$. The space $U\mathcal{D}^n(G)$ is obtained as a quotient of $\mathcal{D}^n(G)$ by the action of the symmetric group Σ_n, which acts by permuting the coordinates of a labeled configuration. Interesting examples among these discretized configuration spaces have been described by Abrams and Ghrist (see [1,2,12]). We include Example 3.3 in this paper illustrating the discrete Morse function defined in the proof of Lemma 2.2.

For a given n, Theorem 2.1 of [1] gives sufficient conditions on G to guarantee that $\mathcal{C}^n(G)$ deformation retracts onto the subspace $\mathcal{D}^n(G)$, a cubical complex. We state the theorem here for reference. An *essential vertex* of a graph is a vertex whose degree is not equal to 2.

Theorem A. *([1]) Let G be a graph with at least n vertices. Then $\mathcal{C}^n(G)$ deformation retracts onto $\mathcal{D}^n(G)$ if*

(A′) *each path connecting distinct essential vertices of G has length at least $n+1$, and*
(B′) *each homotopically essential path connecting a vertex to itself has length at least $n+1$.*