Topology and its Applications 178 (2014) 200–218

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Classifying non-splitting fiber preserving actions on prism manifolds

John Kalliongis [a,*], Ryo Ohashi [b,*]

[a] *Department of Mathematics and Computer Science, Saint Louis University, 220 North Grand Boulevard, St. Louis, MO 63103, USA*

[b] *Department of Mathematics and Computer Science, King's College, 133 North River Street, Wilkes-Barre, PA 18711, USA*

ARTICLE INFO

Article history:
Received 25 September 2013
Received in revised form 17 September 2014
Accepted 17 September 2014
Available online 1 October 2014

MSC:
primary 57M99
secondary 55S37, 57S99

Keywords:
Finite group action
Prism manifold
Equivalence of actions
Orbifold
Isometry

ABSTRACT

In this paper, we classify the finite groups of fiber preserving isometries G which act on a prism manifold $M(b,d)$ and do not leave a Heegaard Klein bottle invariant. We construct the following groups of isometries $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathrm{Dih}(\mathbb{Z}_3)$ and $\mathrm{Dih}(\mathbb{Z}_6)$, which act on $M(b,2)$ preserving the longitudinal fibering and not leaving any fibered Heegaard Klein bottle invariant. For the prism manifold $M(1,d)$ we construct the following groups of isometries $\mathbb{S}_4$, $\mathbb{A}_5$, and $\mathbb{A}_4$, which act on $M(1,d)$ preserving the meridian fibering and not leaving any Heegaard Klein bottle invariant. Let G_0 be the normal subgroup of G consisting of the isometries which leave every fiber invariant. We show that if G acts on $M(b,d)$ preserving the longitudinal fibering and not leaving any fibered Klein bottle invariant, then $M(b,d) = M(b,2)$ and G/G_0 is isomorphic to one of the groups $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathrm{Dih}(\mathbb{Z}_3)$ and $\mathrm{Dih}(\mathbb{Z}_6)$; and if G preserves the meridian fibering not leaving any fibered Klein bottle invariant, then $M(b,d) = M(1,d)$ and G/G_0 is isomorphic to one of the groups $\mathbb{S}_4$, $\mathbb{A}_5$, and $\mathbb{A}_4$. We give complete descriptions of the group G in each of these cases.

0. Introduction

In [5], the authors classify, up to equivalence, the finite group actions which act on a prism manifold and leave a Heegaard Klein bottle invariant. It was found that these actions are completely determined by their restrictions to a Heegaard Klein bottle, and have representatives which are fiber preserving isometries preserving the two distinct fiberings on a prism manifold. In this paper, we classify the finite groups of fiber preserving isometries acting on a prism manifold which do not leave any Heegaard Klein bottle invariant.

We now define a prism manifold. Let V be a solid torus and let W be a twisted I-bundle over a Klein bottle. Each boundary of both V and W is a torus $S^1 \times S^1$. For relatively prime integers b and d, there exist integers a and b such that $ad - bc = -1$. The *prism manifold* $M(b,d)$ is obtained by identifying the

* Corresponding authors.
E-mail addresses: kalliongisje@slu.edu (J. Kalliongis), ryoohashi@kings.edu (R. Ohashi).

http://dx.doi.org/10.1016/j.topol.2014.09.010

boundary of V to the boundary of W by the homeomorphism $\psi: \partial V \to \partial W$ defined by $\psi(u,v) = (u^a v^b, u^c v^d)$ for $(u,v) \in \partial V = S^1 \times S^1$. The integers b and d determine $M(b,d)$, up to homeomorphism. An embedded Klein bottle K in $M(b,d)$ is called a *Heegaard Klein bottle* if for any regular neighborhood $N(K)$, it follows that $N(K)$ is a twisted I-bundle over K and the closure of $M(b,d) - N(K)$ is a solid torus. Any G-action which leaves a Heegaard Klein bottle invariant is said to *split*.

A prism manifold $M(b,d)$ can be fibered in two distinct ways so that the quotient space, the space obtained by identifying fibers to points, is either a 2-sphere $\Sigma(2,2,d)$ with three exceptional points of orders 2, 2 and d, or a projective plane $\mathbb{P}^2(b)$ with one exceptional point of order b, see [3] or [7]. If the quotient space is $\Sigma(2,2,d)$, we say $M(b,d)$ has a *longitudinal fibering*; if the quotient space is a projective plane $\mathbb{P}^2(b)$, we say $M(b,d)$ has a *meridian fibering*. A G-action is *fiber preserving* (or *preserves a fibering*) if every element of G maps fibers to fibers.

Let $\mathbb{S}^3$ be the 3-sphere viewed as the set of quaternions $\{u + vj \mid u, v \in \mathbb{C} \text{ and } |u|^2 + |v|^2 = 1\}$. Many of the computations in this paper use the formulae $uj = j\bar{u}$ and $(u + vj)^{-1} = \bar{u} - vj$. We may view the circle $S^1 \subset \mathbb{S}^3$ as $\{u + 0j \mid u \in \mathbb{C} \text{ and } |u| = 1\}$.

The standard elliptic structure on the 3-sphere $\mathbb{S}^3$ is associated with the orthogonal group $O(4)$ under its action on $\mathbb{S}^3$, and therefore giving $O(4)$ as the group of isometries of $\mathbb{S}^3$ and $SO(4)$ as the orientation preserving subgroup. A 3-orbifold (or 3-manifold) M has an elliptic structure if there exists a finite group of isometries $\Gamma \leq O(4)$ such that there is an orbifold (or 3-manifold) covering $\mathbb{S}^3 \to \mathbb{S}^3/\Gamma = M$. An isometry of M is a homeomorphism of M which lifts to an isometry of $\mathbb{S}^3$. Let $\sigma: \mathbb{S}^3 \times \mathbb{S}^3 \to SO(4)$ be the homomorphism defined by $\sigma(q_1, q_2)(q) = q_1 q q_2^{-1}$. Now σ is onto with kernel $\mathbb{Z}_2 = \langle(-1,-1)\rangle$. For a more complete discussion, see [4] and [6].

Let G be a fiber preserving action on $M(b,d)$ and let G_0 be the normal subgroup of G consisting of the isometries which leave every fiber invariant. If G preserves the longitudinal fibering and does not leave any fibered Klein bottle invariant, then by [5] it follows that $M(b,d) = M(b,2)$ and G/G_0 is isomorphic to one of the groups $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathrm{Dih}(\mathbb{Z}_3)$ or $\mathrm{Dih}(\mathbb{Z}_6)$. If G preserves the meridian fibering and does not leave any fibered Klein bottle invariant, then again by [5] it follows that $M(b,d) = M(1,d)$ and G/G_0 is isomorphic to one of the groups $\mathbb{S}_4$, $\mathbb{A}_5$ or $\mathbb{A}_4$.

In Section 1 we show that the groups $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathrm{Dih}(\mathbb{Z}_3)$ and $\mathrm{Dih}(\mathbb{Z}_6)$, act as a group of isometries on $M(b,2)$ preserving the longitudinal fibering and not leaving any fibered Heegaard Klein bottle invariant. In addition, we show that the groups $\mathbb{S}_4$, $\mathbb{A}_5$ and $\mathbb{A}_4$, act as a group of isometries on $M(1,d)$ preserving the meridian fibering and not leaving any fibered Klein bottle invariant. These groups project to isomorphic actions on there respective quotient spaces. In Section 2 we enumerate all the finite groups of isometries which act on $M(b,2)$ and preserve the longitudinal fibering, and identify the ones which do not leave a Heegaard Klein bottle invariant. Section 3 enumerates all the finite groups of isometries which act on $M(1,d)$ and preserve the meridian fibering, and identify the ones which do not leave a Heegaard Klein bottle invariant.

The authors wish to sincerely thank the referee for a careful reading of the paper, and for making valuable suggestions and corrections to the original manuscript.

1. Examples of non-splitting fiber preserving actions on prism manifolds

In this section we construct some finite group actions on a prism manifold $M(b,d)$ which preserve a fibering, but do not leave any Heegaard Klein bottle invariant. Actions which leave a Heegaard Klein bottle invariant are said to split.

We begin by considering actions which preserve the longitudinal fibering $h_l: M(b,d) \to \Sigma(2,2,d)$, which is induced by the fibering $\mathbb{F}_l = \langle pS^1 \rangle_{p \in \mathbb{S}^3}$ on $\mathbb{S}^3$ with $H_l: \mathbb{S}^3 \to \mathbb{S}^2$ defined by $H_l(u+vj) = u/\bar{v}$. By Theorem 13 in [5], if $\varphi: G \to \mathrm{Diff}(M(b,d))$ is an action which preserve a longitudinal fibering and does not split, then $d = 2$ and G/G_0 is either $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathrm{Dih}(\mathbb{Z}_3)$, or $\mathrm{Dih}(\mathbb{Z}_6)$, where G_0 is the subgroup of G consisting of elements which leave every longitudinal fiber invariant. Hence we consider only the prism manifolds $M(b,2)$. Note