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Preservation and destruction in simple refinements

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A B S T R A C T

If σ is a topology on X and $A \subseteq X$, we let $\langle \sigma, A \rangle$ denote the topology generated by σ and A, i.e., the topology with $\sigma \cup \{A\}$ as a subbasis. Any refinement of a topology obtained like this – by declaring just one new set to be open – we call *simple*. The present paper investigates the preservation of various properties in simple refinements. The locally closed sets (sets open in their closure) play a crucial role here: it turns out that many properties are preserved in a simple refinement by A if and only if A is locally closed. We prove this for the properties of regularity, completely regularity, (complete) metrizability, and (complete) ultrametrizability. We also show that local compactness is preserved in a simple refinement by A if and only if both A and its complement are locally closed.

1. Introduction

Possibly the most widely known result about changing a topology is the following: given a compact Hausdorff topology on a set X, any finer topology on X is non-compact, and any coarser topology is non-Hausdorff. This can be rephrased by saying that compact Hausdorff spaces are "minimally Hausdorff" and "maximally compact". Many other results are also known about spaces that have a certain property maximally or minimally. This has been a lively area of study, and a thorough summary of results like this can be found at the end of [14].

We change the pattern of these results as follows. Instead of looking for particular spaces in which a property P cannot be preserved under refinement, we look at arbitrary spaces satisfying P. In some

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refinements P might be preserved and in others P might be destroyed, and our basic question is: *which ones are which?*

Mostly, we restrict ourselves to a special kind of topological refinement. If σ is a topology on a set X and $A \subseteq X$, we let $\langle \sigma, A\rangle$ denote the topology generated by σ and A, i.e., the topology with $\sigma \cup \{A\}$ as a subbasis. Any refinement of a topology obtained like this – by declaring just one new set to be open – we call **simple**. Given a property P, we will try to determine for which sets A a simple refinement by A preserves or destroys the property P.

The locally closed sets play an important role in these results. A set $A \subseteq X$ is called **locally closed** (with respect to some topology on X) if it satisfies any of the following equivalent properties:

Lemma 1.1. *Let X be a topological space and $A \subseteq X$. The following are equivalent:*

(1) *A is open in its closure.*
(2) *A is the intersection of an open set and a closed set.*
(3) *$A = U \setminus V$ with U and V either both open or both closed.*
(4) *If $x \in A$, there is some open $U \subseteq X$ with $x \in U$ such that $U \cap A$ is closed in U.*

In what follows, we will see that many nice properties of a topology σ are preserved in $\langle \sigma, A\rangle$ if A is locally closed, and are destroyed if A is not locally closed.

We will often need to consider several topologies on a single set. To avoid confusion, we will write $\overline{A}^{\sigma}$ to mean the closure of A with respect to the topology σ, and we will use other similar conventions for other topological operations. If σ is a topology on X and $\mathcal{A}$ is a collection of subsets of X, then $\langle \sigma, \mathcal{A}\rangle$ is the topology with $\sigma \cup \mathcal{A}$ as a subbasis (and $\langle \sigma, A\rangle$ is an abbreviation for $\langle \sigma, \{A\}\rangle$ when $A \subseteq X$). If σ and τ are two topologies on X and $\sigma \subseteq \tau$, then $[\sigma, \tau]$ denotes the set of all topologies α on X such that $\sigma \subseteq \alpha \subseteq \tau$. This notation arises from the fact that we consider $[\sigma, \tau]$ to be an interval in the lattice of all topologies on X. If $B \subseteq X$, we let $\sigma \restriction B = \{U \cap B : U \in \sigma\}$ denote the subspace topology that B inherits from σ.

The following few lemmas are not difficult to prove, and they will help us keep track of what is going on as we move between σ and $\langle \sigma, A\rangle$:

Lemma 1.2. *Let σ and τ be topologies on a set X with $\sigma \subseteq \tau$ and let $B \subseteq X$. Then $\overline{B}^{\tau} \subseteq \overline{B}^{\sigma}$.*

Lemma 1.3. *Let σ be a topology on a set X and let $A \subseteq X$. If $x \notin A$, then $\{U \in \sigma : x \in U\}$ is a neighborhood basis for x in $\langle \sigma, A\rangle$. If $x \in A$ then $\{U \cap A : x \in U \in \sigma\}$ is a neighborhood basis for x in $\langle \sigma, A\rangle$. If $\mathcal{B}$ is a basis for σ, then "σ" can be replaced with "$\mathcal{B}$" in the definitions of these neighborhood bases.*

Proof. This follows directly from the fact that $\sigma \cup \{A\}$ is a subbasis for $\langle \sigma, A\rangle$. □

Lemma 1.4. *Let σ be a topology on a set X and let $A \subseteq X$. If $B \subseteq X$ then $\{U \cap B : U \in \sigma \cup \{A\}\}$ is a subbasis for $\langle \sigma, A\rangle \restriction B$. Moreover, $\langle \sigma, A\rangle \restriction B = \langle \sigma \restriction B, A \cap B\rangle$. In other words, the operations of taking subspaces and taking simple refinements commute.*

Proof. Because $\sigma \cup \{A\}$ is a subbasis for $\langle \sigma, A\rangle$, $\{U \cap B : U \in \sigma \cup \{A\}\}$ is a subbasis for $\langle \sigma, A\rangle \restriction B$. To prove the second claim, note that $\{U \cap B : U \in \sigma \cup \{A\}\} = \{U \cap B : U \in \sigma \cup \{A \cap B\}\}$, and the latter is by definition a subbasis for $\langle \sigma \restriction B, A \cap B\rangle$. □

Lemma 1.5. *Let σ be a topology on a set X and let $A \subseteq X$. If $B \subseteq A$ or $B \subseteq X \setminus A$, then $\sigma \restriction B = \langle \sigma, A\rangle \restriction B$.*

Proof. This is a special case of Lemma 1.4. □