Topology and its Applications 178 (2014) 248–264

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Phantom mappings and a shape-theoretic problem concerning products

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A R T I C L E   I N F O

Article history:
Received 1 October 2013
Accepted 27 September 2014
Available online 10 October 2014

MSC:
54B10
54B35
54C56

Keywords:
Hawaiian earring
Pointed sum of 1-spheres
Cartesian product
Direct product
Shape category
Resolution
Homotopy expansion
Phantom mapping

A B S T R A C T

In the paper one considers the Hawaiian earring $(\mathbb{H}, *)$, the wedge $(\mathbb{P}, *)$ of a sequence of 1-spheres and their Cartesian product $(\mathbb{H} \times \mathbb{P}, *)$. One also considers the shape morphisms $S[\pi_\mathbb{H}]$, $S[\pi_\mathbb{P}]$, induced by the canonical projections $\pi_\mathbb{H}: \mathbb{H} \times \mathbb{P} \to \mathbb{H}$, $\pi_\mathbb{P}: \mathbb{H} \times \mathbb{P} \to \mathbb{P}$. The shape-theoretic problem asks if there exist a polyhedron Z and a shape morphism $H: Z \to \mathbb{H} \times \mathbb{P}$, $H \neq S[*]$, such that $S[\pi_\mathbb{H}]H = S[*]$ and $S[\pi_\mathbb{P}]H = S[*]$. Here $S[*]$ denotes the shape morphisms, induced by the constant mappings $*: Z \to \mathbb{H} \times \mathbb{P}$, $*: Z \to \mathbb{H}$, and $*: Z \to \mathbb{P}$. Answering this problem affirmatively, would imply that the Cartesian product $\mathbb{H} \times \mathbb{P}$ is not a product in the shape category of topological spaces. The main result of the paper establishes equivalence between the shape-theoretic problem and a problem involving phantom mappings.

1. Introduction

1.1. In 1977 Y. Kodama proved that the Cartesian product of an FANR and a paracompact space is a (direct) product in the shape category of topological spaces Sh(Top) [5, Theorem 3′]. In the same paper he stated the following problem, which is still open.

Problem 1. Let X be a metrizable movable compactum and let M be a metrizable space. Is there a product of X and M in the shape category of metrizable spaces Sh(Met)?

A natural candidate for an affirmative answer to Problem 1 is the Cartesian product $X \times M$. Let $\mathbb{H}$ denote the Hawaiian earring and let $\mathbb{M} = P_1 \vee P_2 \vee \dots$ be the wedge (pointed sum) of a sequence of copies

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http://dx.doi.org/10.1016/j.topol.2014.09.011

$(P_i, *)$ of the pointed 1-dimensional sphere $(S^1, *)$, $\mathbb{M}$ endowed with the simplicial metric topology. Then the following open problem is a very special case of Problem 1.

Problem 2. Is the Cartesian product $\mathbb{H} \times \mathbb{M}$ a product in the shape category of metrizable spaces Sh(Met)?

1.2. In general, polyhedra (CW-complexes) endowed with the weak topology are not metrizable spaces. Therefore, we prefer to study the analogue of Problem 2 in the shape category of topological spaces Sh(Top) and stratifiable spaces Sh(Strat).

Let $\mathbb{P} = P_1 \vee P_2 \vee \ldots$ be the wedge (pointed sum) of a sequence of copies $(P_i, *)$ of the pointed 1-dimensional sphere $(S^1, *)$, $\mathbb{P}$ endowed with the weak topology.

Problem 3. Is the Cartesian product $\mathbb{H} \times \mathbb{P}$ a product in the shape category Sh(Top) of topological spaces (Sh(Strat) of stratifiable spaces)?

Note that the Hawaiian earring is the simplest movable continuum, which is not an FANR (see [12, II, §7, Example 3]). In the joint paper with J. Dydak [3], it was shown that the Cartesian product of the dyadic solenoid and $\mathbb{P}$ is not a product in Sh(Top). However, the dyadic solenoid is not movable (see [12, II, §7, Example 2]).

Stratifiable spaces were introduced in 1961 by J. Ceder under the name of M_3-spaces [2]. They were renamed stratifiable by C.J.R. Borges in [1]. Ceder proved that metric spaces and polyhedra (carriers of simplicial complexes endowed with the weak topology) are stratifiable spaces [2, Theorems 2.2, 2.5 and Corollary 8.6]. Moreover, the Cartesian product of two (countably many) stratifiable spaces is a stratifiable space (see [2, Theorem 2.4] or [4, Lemma 2.6]).

The objects of the category Sh(Top) are topological spaces Z. The shape morphisms $F: Z \to W$ are defined using polyhedral expansions of spaces ([12, I, §4 and §6] or [6, II, §8.2]). For homotopic mappings $g, g': Z \to W$, we use the standard notation $g \simeq g'$. The homotopy class of a mapping g is denoted by $[g]$. By H(Top) we denote the category of topological spaces and homotopy classes of mappings. We denote by S: H(Top) $\to$ Sh(Top) the shape functor. It keeps spaces fixed and maps morphisms of H(Top) to the induced shape morphisms (see [12, I, §2.3] or [6, II, §8.2]). Whenever Q is a polyhedron (weak topology) or a simplicial complex endowed with the metric topology, then every shape morphism $G: Z \to Q$ is induced by a unique homotopy class $[g]$ of mappings $g: Z \to Q$, i.e., $G = S[g]$.

When we say that the Cartesian product $X \times P$ of a metric compactum X and a polyhedron P is a product in the shape category Sh(Top) (Sh(Strat)), we mean that, for the canonical projections $\pi_X: X \times P \to X$, $\pi_P: X \times P \to P$ and for every topological (stratifiable) space Z, the following assertions $(\mathrm{ES})_Z$ and $(\mathrm{US})_Z$ hold (E and U in the abbreviations stand for existence and uniqueness in shape, respectively).

$(\mathrm{ES})_Z$ *For every shape morphism $F: Z \to X$ and every homotopy class of mappings $[g]: Z \to P$, there exists a shape morphism $H: Z \to X \times P$ such that $S[\pi_X]H = F$ and $S[\pi_P]H = S[g]$.*

$(\mathrm{US})_Z$ *If $H, H': Z \to X \times P$ are shape morphisms such that $S[\pi_X]H = S[\pi_X]H'$ and $S[\pi_P]H = S[\pi_P]H'$, then $H = H'$.*

In an analogous way we interpret the statement that the Cartesian product $X \times M$ is a product in the shape category Sh(Met).

In a previous paper the author proved the following theorem [9, Theorem 1].

Theorem 1. *For every metric compactum X, every polyhedron P and every topological space Z condition $(\mathrm{ES})_Z$ is fulfilled.*