Topology and its Applications 178 (2014) 276–287

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Fibrations between mapping spaces

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ARTICLE INFO

Article history:
Received 29 July 2014
Accepted 30 September 2014
Available online 11 October 2014

MSC:
primary 55R05
secondary 54C35, 54D50

Keywords:
Mapping space
Hurewicz fibration
Unique path lifting
Dold fibration
Lifting function
Covering projection

ABSTRACT

We study the behaviour of fibre maps under exponentiation, i.e. given a fibration $p: E \to B$ we ask for which spaces X is the induced map between mapping spaces $p_*: E^X \to B^X$ also a fibration. If X is a locally compact space, the positive answer follows easily by the exponential law so in this paper we consider more general spaces and show that the preservation of fibrations is related to the local homotopy properties of the space X. For example, if p is a Dold fibration and X admits a deformation retraction on a compactly generated space, then the induced map p_* is also a Dold fibration. Similar results hold for Hurewicz fibrations with unique path-lifting and for covering spaces, and can be furthermore extended to spaces that admit some numerable cover, whose elements preserve fibration property.

1. Introduction

Given a map $p: E \to B$ and a space X we will denote by E^X and B^X the spaces of continuous maps from X to E and B endowed with the compact-open topology and by $p_*: E^X \to B^X$ the induced map $p_*: f \mapsto p \circ f$. It is often important to know which properties of p are shared by p_*. In this paper we study spaces that *preserve* fibrations, in the sense that if p belongs to some class of fibrations (e.g. Hurewicz fibrations, Dold fibrations or covering spaces) then the induced map p_* belongs to the same class.

1.1. Prior work

The basic case arises when X is locally compact and $p: E \to B$ is a Hurewicz fibration (i.e. it has the homotopy lifting property for arbitrary spaces – cf. [5]). Then we may use the exponential law to transform the homotopy lifting diagram

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[1] The author was partially supported by the Slovenian Research Agency grant P1-02920101.

http://dx.doi.org/10.1016/j.topol.2014.09.014
0166-8641/

$$\begin{array}{ccc} A & \xrightarrow{h} & E^X \\ \big\downarrow & & \big\downarrow p_* \\ A \times I & \xrightarrow[H]{} & B^X \end{array}$$

into the diagram

$$\begin{array}{ccc} A \times X & \xrightarrow{\hat{h}} & E \\ \big\downarrow & \overset{G}{\nearrow} & \big\downarrow p \\ A \times X \times I & \xrightarrow[\widehat{H}]{} & B \end{array}$$

The map G exists by the homotopy lifting property of p, and its adjoint $\widehat{G}: A \times I \to E^X$ yields the lifting of H, which proves that p_* is also a Hurewicz fibration. Smrekar [7,8] was able to extend the above argument to the case where X is any compactly generated space. However, the direct approach breaks down when X is not compactly generated as in that case the exponential law is not available. Of course, a possible way out would be to work in the context of k-spaces (see [5]), but that requires a modification of topologies on products and mapping spaces, and actually answers a different question. In a related work Apery [1] considered covering spaces and showed that they are preserved by contractible spaces, compact CW-complexes and (possibly infinite) graphs.

1.2. Our contribution

All results of this work are derived in one way or the other from Lemma 2.3 which can be roughly stated as follows: if X has a deformation retract that preserves a class of fibrations, then X preserves the same class of fibrations, up to a homotopy. The precise statement and the proof are quite technical, as they involve a roundabout definition of a lifting function, and a delicate proof of its continuity. Lemma 2.3 has various consequences that are thoroughly studied in the second part of the paper, where we apply the lemma to several important classes of fibrations.

The first comprehensive class of fibrations that we consider are Hurewicz fibrations with unique path lifting (see [9]) and we prove Theorem 3.2: if $p: E \to B$ is a Hurewicz fibration with unique path-lifting, and if X admits a numerable open or a locally finite closed cover such that each of its elements can be deformed to a compactly generated subset, then $p_*: E^X \to B^X$ is also a Hurewicz fibration with unique path lifting. In particular, these fibrations are preserved by all locally contractible spaces or more generally, by all spaces that are locally homotopy equivalent to a compactly generated space.

Then we study classical covering spaces, which can be viewed as locally trivial Hurewicz fibrations with discrete fibres. Local triviality and the topology of the fibre are both affected by basic operations on fibrations like compositions, products and construction of mapping spaces (cf. [9]). Nevertheless, in Theorem 3.8 we prove that every (strongly) locally contractible space X with finitely many path-components preserves all covering spaces.

Since our construction of the lifting function is based on a homotopy deformation, it is not surprising that for general fibrations we obtain a lifting only up to a homotopy. That is why a more natural context for our approach is given by Dold fibrations (see [5, Section 1.2]): our main result is Theorem 3.13 stating that Dold fibrations are preserved by every X which admits a finite numerable covering, whose elements can be deformed to subspaces that preserve Dold-fibrations.