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On the admissibility of certain local systems

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ABSTRACT

A rank one local system on the complement of a hyperplane arrangement is said to be admissible if it satisfies certain non-positivity condition at every resonant edges. It is known that the cohomology of admissible local system can be computed combinatorially. In this paper, we study the structure of the set of all non-admissible local systems in the character torus. We prove that the set of non-admissible local systems forms a union of subtori. The relations with characteristic varieties are also discussed.

1. Introduction

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a finite set of hyperplanes in $\mathbb{P}^\ell_{\mathbb{C}}$. The hyperplane arrangement $\mathcal{A}$ determines a poset $L(\mathcal{A})$ of subspaces obtained as intersections of hyperplanes in $\mathcal{A}$. The combinatorial structure of $L(\mathcal{A})$ is deeply related to the topology of $M(\mathcal{A})$. In [13], Orlik and Solomon proved that the cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ can be described in terms of combinatorial structures of $L(\mathcal{A})$. However the homotopy type of $M(\mathcal{A})$ cannot be determined by $L(\mathcal{A})$. Indeed, in [14], Rybnikov proved that the fundamental group $\pi_1(M(\mathcal{A}))$ cannot be determined by $L(\mathcal{A})$. The combinatorial decidability of other topological invariants is still widely open. One of such invariants is rank one local system cohomology groups over $M(\mathcal{A})$, which is originally motivated by hypergeometric functions [1,9]. Rank one local systems are parametrised by points in the character torus $\mathbb{T}(\mathcal{A}) = \mathrm{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*)$. For generic $t \in \mathbb{T}(\mathcal{A})$, it is proved [10] that $H^i(M(\mathcal{A}), \mathcal{L}_t) = 0$ for $i \neq \ell$, where $\mathcal{L}_t$ is a rank one local system corresponding to $t \in \mathbb{T}(\mathcal{A})$. Furthermore,

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Esnault–Schechtman–Viehweg [7] (and [15]) proved that for a local system $\mathcal{L}_t$ such that the residue of associated logarithmic connection at each hyperplane is not a positive integer, the cohomology group $H^i(M(\mathcal{A}), \mathcal{L}_t)$ can be computed by using the cochain complex, so-called the Aomoto complex, defined on the graded module $H^*(M(\mathcal{A}), \mathbb{C})$, see §3. Such a local system is now called an admissible local system (see Definition 3.1). Therefore, for an admissible local system $\mathcal{L}_t$, the cohomology $H^*(M(\mathcal{A}), \mathcal{L}_t)$ is combinatorially computable.

For some arrangements, it has been proved that all rank one local systems are admissible [12]. However, in general, the set of non-admissible local systems is non-empty. A natural strategy to study combinatorial decidability of local system cohomology groups is: (a) Determine the set of all non-admissible local systems in $\mathbb{T}(\mathcal{A})$. (b) Compute the local system cohomology groups for non-admissible local systems. The purpose of this paper is related to the part (a) of the above strategy. We study the basic properties of the set of non-admissible local systems in the character torus $\mathbb{T}(\mathcal{A})$ of a given line arrangement $\mathcal{A}$.

The paper is organised as follows. In §2, we generalise the notion of admissible local systems. Consider the exponential map $\mathrm{Exp}\colon V = \mathbb{C}^n \longrightarrow \mathbb{T} = (\mathbb{C}^*)^n$, with kernel $\Lambda \cong \mathbb{Z}^n$. We introduce the notion "Φ-admissibility" on the algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$, for any finite set $\Phi \subset V^*$ of linear forms which preserve integral structure. We prove that the set of non-admissible points in $\mathbb{T}$ forms a union of subtori. We also give several conditions on $t \in \mathbb{T}$ to be admissible/non-admissible. In §3, we describe the relation between the notions of Φ-admissibility and admissibility, underlining that the latter is a particular case of the former. We then apply results from the previous section to the case of character torus of the complement complex line arrangements. In §4, we discuss the relation between non-admissible local systems and characteristic varieties of line arrangements. In particular, we prove that the local system corresponding to a point in the translated component (in the characteristic variety) is non-admissible. In §5, we describe several examples.

2. General theory

Let $V \cong \mathbb{C}^n$ be a vector space and let $\mathbb{T} \cong (\mathbb{C}^*)^n$ be a complex torus. Consider the exponential mapping

$$\mathrm{Exp}\colon V \longrightarrow \mathbb{T},$$

induced by the usual exponential function $\mathbb{C} \longrightarrow \mathbb{C}^*$, $t \mapsto \mathrm{Exp}(t) = \exp(2\pi i t)$, where $i = \sqrt{-1}$. Let $\Lambda := \ker(\mathrm{Exp})$. Note that Λ is a lattice and hence $\Lambda \cong \mathbb{Z}^n$. Then we have that $V = \Lambda \otimes \mathbb{C}$ and define $V_{\mathbb{R}} = \Lambda \otimes \mathbb{R} \cong \mathbb{R}^n$.

Consider a set $\Phi := \{\alpha_1, \dots, \alpha_r\}$ of linear maps $\alpha_i\colon V \longrightarrow \mathbb{C}$ such that $\alpha_i(\Lambda) \subset \mathbb{Z}$ for all $i = 1, \dots, r$. It is easily seen that there exists a character $\tilde{\alpha}_i\colon \mathbb{T} \longrightarrow \mathbb{C}^*$ such that the diagram (1) is commutative

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{Exp}} & \mathbb{T} \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \tilde{\alpha}_i} \\ \mathbb{C} & \xrightarrow{\mathrm{Exp}} & \mathbb{C}^*. \end{array} \tag{1}$$

Definition 2.1. Consider $t \in \mathbb{T}$. It is called *Φ-admissible* if there exists $v \in V$ such that $\mathrm{Exp}(v) = t$ and $\alpha_i(v) \notin \mathbb{Z}_{>0}$ for all $\alpha_i \in \Phi$.

Example 2.2. The unit $1 \in \mathbb{T}$ is Φ-admissible for all Φ. In fact, $\mathrm{Exp}(0) = 1$ and $\alpha_i(0) = 0 \notin \mathbb{Z}_{>0}$ for all $\alpha_i \in \Phi$.

Note that, for any given Φ, we can write $\mathbb{T} = \mathrm{Adm}(\Phi) \sqcup \mathrm{NonAdm}(\Phi)$, where $\mathrm{Adm}(\Phi)$ is the set of Φ-admissible elements and $\mathrm{NonAdm}(\Phi)$ is the set of non-Φ-admissible elements.