Topology and its Applications 178 (2014) 315–319

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Uniform approximation of homeomorphisms by diffeomorphisms

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ARTICLE INFO

Article history:
Received 22 May 2014
Received in revised form 2 October 2014
Accepted 3 October 2014
Available online 15 October 2014

MSC:
57R12
57Q55
58C35
28D05

Keywords:
Uniform approximation
Homeomorphism
Diffeomorphism
Isotopic
Volume preserving

ABSTRACT

We prove that a compactly supported homeomorphism of a smooth manifold of dimension $n \geq 5$ can be approximated uniformly by compactly supported diffeomorphisms if and only if it is isotopic to a diffeomorphism. If the given homeomorphism is in addition volume preserving, then it can also be approximated uniformly by volume preserving diffeomorphisms.

1. Introduction

A basic problem in differential topology is to detect if a given homeomorphism of a smooth manifold M can be approximated uniformly by diffeomorphisms. This question dates back at least as far as [1] (and [2] in the volume preserving case). An obvious necessary condition is that the homeomorphism is isotopic to the identity. A solution to the problem was found independently by J.R. Munkres [3,4] and M.W. Hirsch [5] in the form of an obstruction theory, and by E.H. Connell [6] for $\mathbb{R}^n$ via an intermediate uniform approximation by PL homeomorphisms. In particular, the above necessary condition is always sufficient if the dimension of M is at most 3 [3], but not necessarily in dimension 4 [7]. In contrast, continuous maps can always be approximated uniformly by smooth maps [8].

In this note, we combine the above results to prove the following theorem.

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http://dx.doi.org/10.1016/j.topol.2014.10.003
0166-8641/

Theorem 1. *Let M be a closed smooth manifold of dimension $n \geq 5$. A homeomorphism φ of M can be approximated uniformly by diffeomorphisms if and only if φ is isotopic to a diffeomorphism.*

See Section 2 for details. We first reduce the theorem to the case of the unit ball in $\mathbb{R}^n$ in Section 3, and then prove the local result in Section 4. Homeomorphisms of non-compact manifolds and manifolds with boundary are discussed in Section 5. Not too surprisingly, the problem remains unsolved in the exceptional dimension $n = 4$.

One can then ask whether a volume preserving homeomorphism that can be approximated by diffeomorphisms can also be approximated uniformly by volume preserving diffeomorphisms. This question was answered in the affirmative independently by Y.-G. Oh [9] and J.-C. Sikorav [10]. It is of particular interest in dimension two in the context of C^0-symplectic topology [9,11–13], which was the starting point of the author's continued interest in this question. Interesting variants of this problem have also been studied, e.g. approximation by almost everywhere diffeomorphisms [2] or in measure (in connection with Lusin's theorem) [14,15].

2. Preliminaries

Let M be a closed smooth manifold of dimension $\dim M \geq 5$. Without loss of generality, we may assume that M is connected. Denote by $\mathrm{Homeo}(M)$ and $\mathrm{Diff}(M)$ the group of homeomorphisms and the subgroup of diffeomorphisms of M, respectively. The group $\mathrm{Homeo}(M)$ is equipped with the uniform (or compact-open) topology. It is metrizable by the uniform distance of homeomorphisms or by the so-called C^0-distance (that is, the uniform distance of homeomorphisms and of their inverses). Only the latter is complete. However, if a sequence of homeomorphisms converges uniformly to another homeomorphism (in other words, if a limit exists), then it also converges in the C^0-metric.

Recall that any C^1-diffeomorphism can be approximated uniformly by C^∞-diffeomorphisms [8, Theorem 2.7]. Thus for the purposes of this paper, we do not need to distinguish between diffeomorphisms that are of class C^1 or C^∞.

An isotopy is a continuous map $\Phi\colon [0,1] \times M \to M$ such that $\varphi_t = \Phi(t, \cdot)$ is a homeomorphism for each t; two homeomorphisms φ and ψ are said to be isotopic if there exists an isotopy Φ with $\Phi(0, \cdot) = \varphi$ and $\Phi(1, \cdot) = \psi$.

3. From local to global

The following main lemma is a local version of Theorem 1, and is due to Munkres [3], Connell [6], and R.H. Bing [16].

Lemma 2 *(Munkres, Connell, Bing). Let $n \geq 5$, and $\varphi\colon B^n \to B^n$ be a homeomorphism of the open unit ball in $\mathbb{R}^n$ that is the identity near the boundary of B^n, and is isotopic to the identity through an isotopy that fixes pointwise a neighborhood of the boundary. Then φ can be approximated uniformly by diffeomorphisms of B^n that are the identity near the boundary.*

A proof will be given in the next section. The only subtlety that is not stated explicitly in the cited references is that the diffeomorphisms that approximate the given homeomorphism can be chosen to be equal to the identity near the boundary, and therefore extend to diffeomorphisms of an ambient manifold. Aside from a few well-known classical theorems in (differential) topology, this last fact is the main ingredient in our proof of Theorem 1.

Assuming the main lemma, we will give a proof of Theorem 1.