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On symmetrical cliquishness and quasi-continuity of functions of two variables

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ABSTRACT

We study the properties of joint cliquishness and quasi-continuity for functions of two variables. We introduce some properties (B) and (C) of functions of two variables such that (B) is an essential weakening of quasi-continuity and (C) is valid, in particular, for functions taking values in separable metrizable spaces. In particular, we prove the following theorem. Let X be a Baire space, Y a topological space which has a countable pseudo-base, Z a metric space and $f : X \times Y \to Z$ a function. Then a residual subset A of X such that f is symmetrically cliquish (quasi-continuous) with respect to x at each point of $A \times Y$ exists if and only if $\{x \in X : f^x$ is cliquish (quasi-continuous)$\}$ is residual in X and conditions (B) and (C) hold.

1. Basic definitions and concepts

For a subset A of a topological space denote by $int\, A$ and $\overline{A}$ the closure and the interior of A, respectively. Let X and Y be topological spaces and (Z, d) a metric space. Recall that for non-empty subsets A of a set X and a function $f : X \to Z$ the number $\omega_f(A) = \sup_{u,v\in A} d(f(u), f(v))$ is called the *oscillation of f on A*. We consider functions with values in a metric space only. So, we give definitions for such functions. A function $f : X \to Z$ is said to be *quasi-continuous* [5] *(cliquish* [9]*) at a point $x \in X$* if for any $\varepsilon > 0$ and a neighborhood U of $x \in X$ there exists a non-empty open subset G of X such that $G \subseteq U$ and $\omega_f(\{x\} \cup G) < \varepsilon$ $(\omega_f(G) < \varepsilon)$. A function $f : X \times Y \to Z$ is said to be *symmetrically quasi-continuous (cliquish) with respect to x at a point $p_0 = (x_0, y_0) \in X \times Y$* if for any $\varepsilon > 0$, neighborhoods U and V of x_0 and y_0 respectively there are a neighborhood G of x_0 and a non-empty open subset H of Y such that $G \times H \subseteq U \times V$ and $\omega_f(\{p_0\} \cup (G \times H)) < \varepsilon$ $(\omega_f(G \times H) < \varepsilon)$. A function $f : X \times Y \to Z$ is said to be *horizontally quasi-continuous* [6] *at a point $p = (x, y) \in X \times Y$* if for any $\varepsilon > 0$, any neighborhoods U

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and V of x and y respectively there are a neighborhood G of x and a point $b \in V$ such that $G \subseteq U$ and $\omega_f(\{p\} \cup (G \times \{b\})) < \varepsilon$. A function f is said to be *quasi-continuous, cliquish, horizontally quasi-continuous, symmetrically quasi-continuous or symmetrically cliquish with respect to* x if it is so at each point.

For a function $f : X \times Y \to Z$ and points $x \in X$, $y \in Y$ the sections $f^x : Y \to Z$ and $f_y : X \to Z$ are defined by $f^x(y) = f_y(x) = f(x, y)$. We say that a function $f : X \times Y \to Z$ *satisfies condition (B)* if for any $\beta > 0$, non-empty open sets U in X and V in Y and a dense in U set $E \subseteq X$ with $\omega_f(E \times V) < \beta$ there exist non-empty open subsets G of X and H of Y such that $G \subseteq U$, $H \subseteq V$ and $\omega_f(G \times H) < \beta$. A function $f : X \times Y \to Z$ *satisfies condition (C)* if for any $\varepsilon > 0$, a non-meagre set E in X and a non-empty open set V in Y there exist a somewhere dense set E_1 in X and a map $g : E_1 \to V$ such that $E_1 \subseteq E$ and $\omega_f(Gr(g)) < \varepsilon$, where $Gr(g) = \{(x, g(x)) : x \in E_1\}$ is the graph of g.

2. Historical background

Investigation of the relationship between separate and joint continuity, which began its history in the classical works of Baire and Osgood, has spread on the various weakening of continuity. The cliquishness [1] is one such weakening.

Unlike of the separate continuity the separate cliquishness of the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ does not guarantee the existence of points of jointly cliquishness [3, Example 4]. However L. Fudali in [3] established that if X is a Baire space, Y is a topological space which has a countable pseudo-base, Z is a metric space, a function $f : X \times Y \to Z$ is quasi-continuous with respect to the first variable and cliquish with respect to the second variable, then f is jointly cliquish.

In [7] it was obtained a result that is similar to the result of Fudali: if X and Y are topological spaces which have a countable pseudo-base, Y is a Baire space, Z is metric space and a function $f : X \times Y \to Z$ is cliquish with respect to the first variable and quasi-continuous with respect to the second variable then there is a residual subset A of X such that f is symmetrically quasi-continuous with respect to x at each point of $A \times Y$.

In [2] K. Bögel introduces a property of functions defined on the product of parallelepipeds. Later in [8] this property was transferred to the case of arbitrary topological spaces and called horizontally quasi-continuity. The horizontally quasi-continuity is weakening of quasi-continuity of a function with respect to the first variable. In proofs of many theorems, in which horizontally quasi-continuity appears, actually used the weaker condition that in [8] called *weak horizontally quasi-continuity.* A map $f : X \times Y \to Z$ is called *weak horizontally quasi-continuous* [8] if for arbitrary open subsets U of X and V of Y and a set $A \subseteq X$ with $U \subseteq \overline{A}$ have that $f(U \times V) \subseteq \overline{f(A \times V)}$. In [1], using the notion of weak horizontally quasi-continuity that there was called lower X-quasi-continuity, A. Bouziad and J.P. Troallic obtained a result that here is served in a slightly simplified form.

Theorem. *(A. Bouziad and J.P. Troallic [1].) Let X be a topological space, Y a topological space which has a countable pseudo-base, Z a metric space and $f : X \times Y \to Z$ a weakly horizontally quasi-continuous function. Then:*

1) *if $\{x \in X : f^x$ is cliquish$\}$ is a residual set in X then there is a residual set A in X such that f is symmetrically cliquish with respect to x at each point of the set $A \times Y$;*
2) *if $\{x \in X : f^x$ is quasi-continuous$\}$ is a residual set in X then there is a residual set A in X such that f is symmetrically quasi-continuous with respect to x at each point of the set $A \times Y$.*

In this paper we will show that, when replaced in Bouziad–Troallic's theorem weak horizontal quasi-continuity at of the conditions (B) and (C), we will strengthen not only this result, but we will get the characterization of the jointly cliquishness of two variables.