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Remainders in compactifications of semitopological and paratopological groups ☆

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ABSTRACT

In this paper, the remainders of semitopological and paratopological groups are investigated. We mainly establish that if G is a non-locally compact semitopological group and bG is a compactification of G such that $Y = bG \setminus G$ has locally a point-countable base, then bG is separable and metrizable. This gives a positive answer to a question posed in Wang and He (2014) [25]. We also show that if G is a non-locally compact $\mathbb{R}_1$-factorizable paratopological group and $Y = bG \setminus G$ is a local $\aleph$-space, then bG is separable and metrizable. Some questions in [14] are answered.

1. Introduction

"A space" in this paper stands for a Tychonoff topological space. A remainder of a space X is the space $bX \setminus X$, where bX is a Hausdorff compactification of X.

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The question when a space X has a Hausdorff compactification with the remainder belonging to a given class of spaces is important. A famous classical result in this direction is the theorem of M. Henriksen and J. Isbell [13]:

Theorem 1.1. *A space X is of countable type if and only if the remainder in any (in some) compactification of X is Lindelöf.*

Recall that a space X is of *countable type* if every compact subspace F of X is contained in a compact subspace $K \subseteq X$ with a countable base of open neighborhoods in X.

Recall that a *semitopological group* (resp., *paratopological group*) is a group with a topology such that the multiplication in the group is separately continuous (resp., jointly continuous). If G is a paratopological group and the inverse operation of G is continuous, then G is called a *topological group*. The reader can find a lot of recent progress about paratopological (or semitopological) groups in the survey article [24].

A series of results on remainders of topological groups have been obtained in [2,4,6,7,17]. They show that remainders of topological groups are much more sensitive to the topological properties of groups than the remainders of topological spaces are in general. However, much less is known about remainders of paratopological (semitopological) groups [24]. The reader can find some recent progress in this direction in [9,14,18,25–27]. In this paper, we will continue to study how the generalized metrizability of remainders affects the paratopological (semitopological) groups.

First, we recall some concepts [1,12].

A base $\mathcal{B}$ for a space X is said to be *uniform* if for each injective sequence $(B_n) \subseteq \mathcal{B}$ and every $x \in \bigcap_{n\in\omega} B_n$, the sequence (B_n) is a base at x.

A base $\mathcal{B}$ for a space X is said to be *weakly uniform* if for each countably infinite family $\mathcal{U} \subseteq \mathcal{B}$ and for each $x \in X$, if $x \in U$ for each $U \in \mathcal{U}$, then $\{x\} = \bigcap \mathcal{U}$.

A base $\mathcal{B}$ for a space X is said to be *sharp* if for every $x \in X$ and every sequence (U_n) of pairwise distinct elements of $\mathcal{B}$ with $x \in U_n$ for all $n \in \omega$, the collection $\{\bigcap_{i\le n} U_i : n \in \omega\}$ forms a base at x.

Recall that a space X has a *base of countable order* (BCO) if X has a base $\mathcal{B}$ such that whenever $x \in X$ and a strictly decreasing sequence (B_n) of elements of $\mathcal{B}$ is such that $x \in \bigcap_{n\in\omega} B_n$, then (B_n) is a base at x.

Let $(\mathcal{U}_n)$ be a sequence of open covers of a space X. Recall that, for every $x \in X$ and n, $\operatorname{st}(x, \mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n : x \in U\}$.

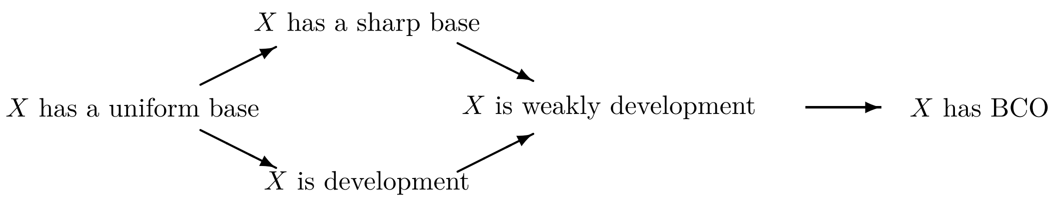
A sequence of open covers $(\mathcal{U}_n)$ of a space X is called:

• A *G_δ-diagonal sequence*, if for every $x \in X$, $\bigcap_{n\in\omega} \operatorname{st}(x, \mathcal{U}_n) = \{x\}$. A space with a G_δ-diagonal sequence is called a space with a *G_δ-diagonal*.

• A *weak development*, if for every $x \in X$ and the sequence (U_n) such that $x \in U_n \in \mathcal{U}_n$ for every n, the sequence $(\bigcap_{i\le n} U_i)$ is a base at x. A space with a weak development is called a *weakly developable space*.

• A *development*, if for every $x \in X$, the sequence $(\operatorname{st}(x, \mathcal{U}_n))$ is a base at x. A space with a development is called a *developable space*.

The implications of the following diagram have been established in [1, Theorem 3.5].



If X is metacompact, then all the five assertions above are equivalent [1, Theorem 3.5].

The question whether a non-locally compact topological group G is separable and metrizable if G has a BCO remainder is still open [17, Question 14]. Arhangel'skiĭ [6] proved that if the remainder of the