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Indecomposable continua in exponential dynamics — Hausdorff dimension

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ABSTRACT

We study some forward invariant sets appearing in the dynamics of the exponential family. We prove that the Hausdorff dimension of the sets under consideration is not larger than 1. This allows us to prove, as a consequence, a result for some dynamically defined indecomposable continua which appear in the dynamics of the exponential family. We prove that the Hausdorff dimension of these continua is equal to one.

1. Introduction

The dynamics of transcendental meromorphic functions has been developed as a counterpart to the theory of rational iterations. A model family $\lambda \mapsto \lambda \exp(z)$ has been a subject of a particular interest, playing a similar role as the family of quadratic polynomials $z^2 + c$ in the iterations of rational maps, see e.g. [2] for a review of results on this family.

There are several properties of the iterations of transcendental entire (and, more generally, transcendental meromorphic) maps which have no counterpart in the dynamics of rational maps.

Below, we recall basic definitions and facts.

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. As usual, we shall denote by f^n the n-th iterate of f: $f^n = f \circ f \circ \cdots \circ f$. The sequence $\{f^n(z)\}_{n=0}^{\infty}$ is called the trajectory of z. The Fatou set $F(f)$ consists of

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all points $z \in \mathbb{C}$ for which there exists a neighbourhood $U \ni z$ such that the family of iterates $f^n_{|U}$ forms a normal family. The complement of $F(f)$ is called the Julia set of f and it is denoted by $J(f)$. An intuitive characterisation of the Julia set says that it carries the chaotic part of the dynamics. See e.g. [1] and [13] for a detailed presentation of the theory.

For the particular family of maps $f_\lambda(z) = \lambda e^z$ the structure of the Julia set and the dynamics have been studied intensively. In 1981 M. Misiurewicz answered the old question, formulated by Fatou, proving that the Julia set of the map $f_1(z) = e^z$ is the whole plane [15]. A striking result, proved independently by M. Lyubich [14] and M. Rees [16], says that, nevertheless, the map is not ergodic with respect to the two-dimensional Lebesgue measure, and for Lebesgue almost every point z the ω-limit set of z (i.e. the set of accumulation points of the trajectory) is just the trajectory of the singular value 0 plus the point at infinity.

On the other hand, there is an open set of parameters λ for which there exists a periodic attracting orbit (i.e. a point p_λ such that $f^k_\lambda(p_\lambda) = p_\lambda$ and $|(f^k_\lambda)'(p_\lambda)| < 1$ for some $k \in \mathbb{N}$). In this case, the Fatou set is nonempty and contains a neighbourhood of this periodic attracting orbit. Actually, it turns out that in this case the Fatou set is just the basin of attraction of this periodic orbit, i.e.

$$F(f_\lambda) = \Big\{ z \in \mathbb{C} : \lim_{n\to\infty} f^{nk}_\lambda(z) = f^i(p_\lambda) \text{ for some } i \in \{0, 1, \dots, k-1\} \Big\}.$$

Moreover, this basin of attraction must contain the whole trajectory of the singular value 0. For instance, if $\lambda \in (0, \frac{1}{e})$, then f_λ has a real attracting fixed point p_λ. For these values of λ the Fatou set of f_λ is connected and simply connected and its complement $J(f_\lambda)$ is a "Cantor bouquet" of curves. See [2] or [20] for a survey of results in this direction.

A general classification theorem for the components of the Fatou set leads to the following corollary: if the parameter λ is chosen so that $f^n_\lambda(0) \to \infty$, then the Julia set of f_λ is the whole plane: $J(f_\lambda) = \mathbb{C}$ (see [7] or [10]).

In this paper, we shall always assume that the parameter λ is chosen so that $f^n_\lambda(0) \to \infty$ sufficiently fast. Thus, in particular, for all maps f_λ considered in this paper we have $J(f_\lambda) = \mathbb{C}$ (see Section 2 for precise assumptions).

We shall explain now the motivation of the present work. It is known that the set of escaping points

$$I(f_\lambda) = \big\{ z : f^n_\lambda(z) \to \infty \big\}$$

can be described in terms of so-called "dynamic rays" (see [21] and [18]). Each dynamic ray is a curve $g_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$. Moreover, $\mathrm{Re}(g_{\underline{s}}(t)) \to +\infty$ as $t \to +\infty$. Each such curve is characterised by an infinite sequence $\underline{s} = (s_0, s_1, \dots)$ of integers. The sequence $\underline{s}$ is frequently called the "external address" of the curve $g_{\underline{s}}$. The dynamical meaning of the sequence $\underline{s}$ is the following: Let us divide the plane $\mathbb{C}$ into horizontal strips

$$P_k = \big\{ z \in \mathbb{C} : (2k-1)\pi - \mathrm{Arg}(\lambda) < \mathrm{Im}(z) \le (2k+1)\pi - \mathrm{Arg}(\lambda) \big\}. \tag{1.1}$$

Observe that every set P_k is mapped bijectively onto $\mathbb{C} \setminus \{0\}$. And the image of the boundary of P_k is the negative real axis.

If z is a point on the curve $g_{\underline{s}}$, $z = g_{\underline{s}}(t)$ with t sufficiently large, then, for every $n \ge 0$, $f^n_\lambda(z) \in P_{s_n}$. The classification of escaping points (see [21], Corollary 6.9) says that this family of curves almost exhausts the set $I(f_\lambda)$. Namely, if $z \in I(f_\lambda)$ then either z belongs to some curve $g_{\underline{s}}$, or z is a landing point of some curve $g_{\underline{s}}$, or else the singular value 0 escapes, $0 = g_{\underline{s}}(t_0)$ for some $t_0 > t_{\underline{s}}$, and z is eventually mapped to the initial piece of the curve $g_{\underline{s}}$, cut off by the point 0, i.e. $f^n_\lambda(z) = g_{\underline{s}}(t')$ for some $t_{\underline{s}} < t' < t_0$.

As an example, let us consider $\lambda = 1$ and the dynamic ray corresponding to the sequence $\underline{s} = (0, 0, 0, \dots)$. This set is just the real line $\mathbb{R}$. The set