Topology and its Applications 178 (2014) 417–437

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The first homology group of the mapping class group of a nonorientable surface with twisted coefficients

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ARTICLE INFO

Article history:
Received 28 April 2014
Received in revised form 22 October 2014
Accepted 22 October 2014
Available online 31 October 2014

MSC:
20F38
57N05
57M99

Keywords:
Mapping class group
Nonorientable surface
Homology of groups

ABSTRACT

We determine the first homology group of the mapping class group $\mathcal{M}(N)$ of a nonorientable surface N with coefficients in $H_1(N;\mathbb{Z})$.

1. Introduction

Let $N_{g,s}$ be a smooth, nonorientable, compact surface of genus g with s boundary components. If s is zero, then we omit it from the notation. If we do not want to emphasise the numbers g, s, we simply write N for a surface $N_{g,s}$. Recall that N_g is a connected sum of g projective planes and $N_{g,s}$ is obtained from N_g by removing s open discs.

Let $\mathrm{Diff}(N)$ be the group of all diffeomorphisms $h\colon N \to N$ such that h is the identity on each boundary component. By $\mathcal{M}(N)$ we denote the quotient group of $\mathrm{Diff}(N)$ by the subgroup consisting of maps isotopic to the identity, where we assume that isotopies are the identity on each boundary component. $\mathcal{M}(N)$ is called the *mapping class group* of N.

The mapping class group $\mathcal{M}(S_{g,s})$ of an orientable surface is defined analogously, but we consider only orientation preserving maps.

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[1] Supported by NCN grant 2012/05/B/ST1/02171.

http://dx.doi.org/10.1016/j.topol.2014.10.015
0166-8641/

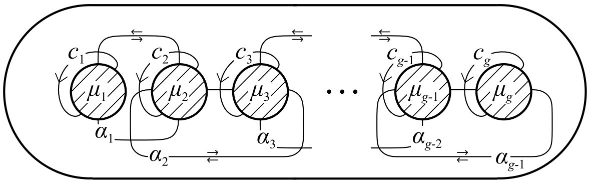

Fig. 1. Surface N_g $[N_{g,1}]$ as a sphere [disc] with crosscaps.

1.1. Background

Homological computations play a prominent role in the theory of mapping class groups. In the orientable case, Mumford [9] observed that $H_1(\mathcal{M}(S_g))$ is a quotient of $\mathbb{Z}_{10}$. Then Birman [1,2] showed that if $g \geq 3$ then $H_1(\mathcal{M}(S_g))$ is a quotient of $\mathbb{Z}_2$, and Powell [11] showed that in fact $H_1(\mathcal{M}(S_g))$ is trivial if $g \geq 3$. As for higher homology groups, Harer [3,4] computed $H_i(\mathcal{M}(S_g))$ for $i = 2, 3$ and Madsen and Weiss [7] determined the integral cohomology ring of the stable mapping class group.

In the nonorientable case, Korkmaz [5,6] computed $H_1(\mathcal{M}(N_g))$ for a closed surface N_g (possibly with marked points). This computation was later [12] extended to the case of a surface with boundary. As for higher homology groups, Wahl [16] identified the stable rational cohomology of $\mathcal{M}(N)$.

As for twisted coefficients, Morita [8] proved that

$$H_1\big(\mathcal{M}(S_g); H_1(S_g;\mathbb{Z})\big) \cong \mathbb{Z}_{2g-2}, \quad \text{for } g \geq 2.$$

There are also similar computations for the hyperelliptic mapping class groups. Tanaka [15] showed that $H_1(\mathcal{M}^h(S_g); H_1(S_g;\mathbb{Z})) \cong \mathbb{Z}_2$ for $g \geq 2$ and in the nonorientable case we showed in [14] that

$$H_1\big(\mathcal{M}^h(N_g); H_1(N_g;\mathbb{Z})\big) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \text{for } g \geq 3.$$

1.2. Main results

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *If $N_{g,s}$ is a nonorientable surface of genus $g \geq 3$ with $s \leq 1$ boundary components, then*

$$H_1\big(\mathcal{M}(N_{g,s}); H_1(N_{g,s};\mathbb{Z})\big) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g \in \{3,4,5,6\} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } g \geq 7. \end{cases}$$

The starting point for this computation is the presentation for the mapping class group $\mathcal{M}(N_{g,s})$, where $g + s \geq 3$ and $s \in \{0, 1\}$, obtained recently by Paris and Szepietowski [10] (Theorems 2.1 and 2.2).

2. Preliminaries

2.1. Nonorientable surfaces

Represent surfaces $N_{g,0}$ and $N_{g,1}$ as respectively a sphere or a disc with g crosscaps and let $\alpha_1, \ldots, \alpha_{g-1}$, $\beta_1, \ldots, \beta_{\frac{g-2}{2}}$ be two-sided circles indicated in Figs. 1 and 2. Small arrows in these figures indicate directions of Dehn twists $a_1, \ldots, a_{g-1}$, $b_1, \ldots, b_{\frac{g-2}{2}}$ associated with these circles.

For any two consecutive crosscaps μ_i, μ_{i+1} we define a *crosscap transposition* u_i to be the map which interchanges these two crosscaps (see Fig. 3). Moreover, if N_g is closed, then we define

$$\varrho = a_1 a_2 \cdots a_{g-1} u_{g-1} \cdots u_2 u_1.$$