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Convergence in van der Waerden and Hindman spaces

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ABSTRACT

We consider four classes of topological spaces which are defined with the aid of convergence with respect to ideals on $\mathbb{N}$. All these classes are subclasses of countably compact spaces, and two of them are also subclasses of sequentially compact spaces. In the first part of the paper (Sections 1 and 2) we prove some properties of these classes. In the second part of the paper (Sections 3 and 4) we focus on spaces defined by two particular ideals connected with well known theorems in combinatorics, namely van der Waerden's theorem and Hindman's theorem. The main aim of this part of the paper is to show that two classes of the considered spaces coincide for the van der Waerden ideal and Hindman ideal respectively.

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1. BW-spaces

Throughout this paper, all topological spaces are assumed to be Hausdorff.

By an *ideal on a set* X we mean a nonempty family of subsets of X closed under taking finite unions and subsets of its elements. By $\mathrm{Fin}(X)$ we denote the ideal of all finite subsets of X (for $X = \mathbb{N}$ we write

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Fin = Fin($\mathbb{N}$)). Moreover, we always assume that ideals are *proper* (i.e. $X \notin \mathcal{I}$) and contains all finite subsets of X (i.e. $\mathrm{Fin}(X) \subseteq \mathcal{I}$). We say that an ideal $\mathcal{I}$ is a *P-ideal* if for every countable family $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for any $n \in \mathbb{N}$.

By Y^X we denote the set of all functions from X into Y, and in the case of $X \subseteq \mathbb{N}$ the set Y^X is the set of all sequences $(y_n)_{n\in X}$ with $y_n \in Y$ for $n \in X$.

Let X be a topological space, $\mathcal{I}$ be an ideal on $\mathbb{N}$ and $A \subseteq \mathbb{N}$. We say that a sequence $(x_n)_{n\in A} \in X^A$ is *$\mathcal{I}$-convergent* to $x \in X$ if

$$\{n \in A : x_n \notin U\} \in \mathcal{I}$$

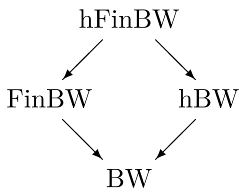
for every open neighborhood $U \subseteq X$, $x \in U$. For $\mathcal{I} = \mathrm{Fin}$ we obtain the ordinary convergence.

We say that the pair $(X, \mathcal{I})$ has

(1) *BW property* if every sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ has an $\mathcal{I}$-convergent subsequence $(x_n)_{n\in A}$ with $A \notin \mathcal{I}$;
(2) *FinBW property* if every sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ has a Fin-convergent subsequence $(x_n)_{n\in A}$ with $A \notin \mathcal{I}$;
(3) *hBW property* if every sequence $(x_n)_{n\in A} \in X^A$ with $A \notin \mathcal{I}$ has an $\mathcal{I}$-convergent subsequence $(x_n)_{n\in B}$ with $B \subseteq A$ and $B \notin \mathcal{I}$;
(4) *hFinBW property* if every sequence $(x_n)_{n\in A} \in X^A$ with $A \notin \mathcal{I}$ has a Fin-convergent subsequence $(x_n)_{n\in B}$ with $B \subseteq A$ and $B \notin \mathcal{I}$.

We write $(X, \mathcal{I}) \in \mathrm{BW}$ if the pair $(X, \mathcal{I})$ has the BW property; we say that an ideal $\mathcal{I}$ has the *BW property* ($\mathcal{I} \in \mathrm{BW}$, in short) if the pair $([0, 1], \mathcal{I}) \in \mathrm{BW}$ (and the same for properties FinBW, hBW and hFinBW). For examples and properties of ideals with(out) the BW-like properties see [6] where these definitions were introduced.

Clearly, the following diagram holds. By "$\mathcal{A} \to \mathcal{B}$" we mean "if $(X, \mathcal{I}) \in \mathcal{A}$ then $(X, \mathcal{I}) \in \mathcal{B}$". If there is no arrow in some direction between $\mathcal{A}$ and $\mathcal{B}$, then it means that, in general, $(X, \mathcal{I}) \in \mathcal{A}$ does not imply $(X, \mathcal{I}) \in \mathcal{B}$ (see [6] for examples of ideals showing that no other implications hold).



The following theorems give us plenty of examples of spaces and ideals with BW-like properties.

Theorem 1.1. *([7, Corollary 5.7]) Suppose that a topological space X satisfies the following condition:*

$$\text{the closure of every countable set in } X \text{ is compact and first countable.} \tag{$*$}$$

If

(1) *$\mathcal{I}$ can be extended to an F_σ ideal (e.g. $\mathcal{I}$ is an analytic P-ideal with the BW property), or*
(2) *$\mathcal{I}$ can be extended to a maximal P-ideal, or*
(3) *$\mathcal{I}$ is a maximal ideal,*

then $(X, \mathcal{I}) \in BW$ (in the case (1) and (2), $(X, \mathcal{I}) \in FinBW$).

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