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Open-constructible functions

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ABSTRACT

We prove that if a continuous one-to-one function between subspaces X, Y of the Cantor set $\mathbf{C}$ maps each open set into constructible or, more general, into resolvable one, then f is a piecewise homeomorphism; i.e., X admits a countable cover $\mathcal{C}$ consisting of pairwise disjoint closed sets, such that for each $C \in \mathcal{C}$ the restriction $f|C$ is a homeomorphism.

1. Introduction

The present paper continues the series of recent publications about open-Borel functions [1,3,4,7,9] – see also [2], where functions of such type are the main subject.

Recall, that a subset E of a metric space X is *resolvable* [5] if for each nonempty closed in X subset F we have

$$cl_X(F \cap E) \cap cl_X(F \setminus E) \neq F$$

If $E \subset X$ is resolvable, then E is Δ^0_2-set in X and vice versa if the space X is Polish. Every constructible[1] set is resolvable [5].

Recall that a function f is open if it maps open sets into open ones.

[1] Recall that *constructible* sets are finite unions of locally closed sets (a set is *locally closed* if it is the intersection of an open set and a closed set).

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More generally, a function f is said to be *open-resolvable* if f maps open sets into resolvable ones.[2]

In the following definitions we will suppose that X is a subspace of the Cantor set $\mathbf{C}$.

A function $f : X \to Y$ is called *piecewise open* if X admits a countable, closed and disjoint cover $\mathcal{C}$, such that for each $C \in \mathcal{C}$ the restriction $f|C$ is open.[3]

A piecewise open function $f : X \to Y$ is called *scatteredly open* if, in addition, the cover $\mathcal{C}$ is scattered, that is:

> for every nonempty subfamily $\mathcal{T} \subset \mathcal{C}$ there is a clopen set $G \subset X$ s.t. $\mathcal{T}_G = \{T \in \mathcal{T} : T \subset G\}$ is a singleton and $T \cap G = \emptyset$ for every $T \in \mathcal{T} \setminus \mathcal{T}_G$.

We established recently that the simplest open-resolvable functions are piecewise open [9, Proposition 3.3].

The following theorem gives a similar result for open-resolvable bijections:

Theorem 1. *Let X be subspaces of the Cantor set $\mathbf{C}$, and $f : X \to Y$ a continuous bijection.*

If the image under f of every open set in X is resolvable in Y, then f is scatteredly open, and hence f is scattered homeomorphism.

This implies obviously the following corollary:

Corollary 1. *Let $f : X \to Y$ be continuous bijection between Polish spaces.*

If the image under f of every open set in X is Δ^2_0-set in Y, then f is scattered homeomorphism.

Theorem 1 gives an affirmative answer to the question [8] in case of bijections whether every continuous open-constructible function between Polish spaces $X, Y \subset \mathbf{C}$ is piecewise open.

In case of surjections we obtain the following theorem:

Theorem 2. *If a continuous function $f : X \to Y$ between $X, Y \subset \mathbf{C}$ maps discrete subspaces in X into resolvable, then f is scatteredly open.*

2. Decomposition of functions into scatteredly and nowhere open ones

Given a function $f : X \to Y$, we shall construct in the next Lemma 1 a subset $Z \subset X$ s.t. the restriction $f|Z$ is *nowhere open* on Z; i.e. for every clopen in X subset U the restriction $f|(U \cap Z)$ is not open.

Lemma 1. *Let X be subspaces of the Cantor set $\mathbf{C}$, and $f : X \to Y$ be a surjection.*

Then there is a closed subset $Z \subset X$ such that the restriction $f|Z$ is nowhere open on Z and the restriction $f|(X \setminus Z)$ is scatteredly open.

Proof. Let us begin by proving the first part of the assertion from lemma stating that for some Z the restriction $f|Z$ is nowhere open on Z.

Indeed, if for some nonempty clopen set $V \subset X$ the restriction $f|V$ is open, then we could construct the closed set

$$X_1 = X \setminus V$$

and the corresponding restriction

[2] Or resolvable, according to Su Gao and V. Kieftenbeld [1].

[3] The symbol $f|C$ means the map $f|C : C \to f(C)$.