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Paratopological groups with a G_δ-diagonal of infinite rank

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We present some classes of paratopological groups with a G_δ-diagonal of infinite rank. In particular, we show that every Hausdorff paratopological group with countable π-character has a G_δ-diagonal of infinite rank, which answers a question posed by A. Arhangel'skii and A. Bella. Also, we construct a Hausdorff paratopological group with uncountable π-character which contains a second countable dense subgroup.

1. Introduction

A *paratopological* (*semitopological*) group is a group endowed with a topology for which multiplication in the group is jointly (separately) continuous. If, additionally, the inversion in a paratopological group is continuous, then it is called a *topological group*.

In 1980, Arhangel'skii showed that every Hausdorff topological group of countable pseudocharacter is submetrizable, i.e., admits a weaker metrizable topology (see [1]). It is easy to see that each submetrizable space has a G_δ-diagonal of infinite rank. Therefore, a Hausdorff topological group G has a G_δ-diagonal of infinite rank if and only if G has countable pseudocharacter.

In this paper, we study when a paratopological group has a G_δ-diagonal of infinite rank. In [7], C. Liu showed that every first countable Hausdorff paratopological group has a regular G_δ-diagonal. After,

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A. Arhangel'skii and A. Bella proved that every Hausdorff first countable paratopological group has a G_δ-diagonal of infinite rank (see [2]). This improves C. Liu's result, since every space with a G_δ-diagonal of rank 3 has a regular G_δ-diagonal (see [2, Proposition 1]).

In [2], A. Arhangel'skii and A. Bella posed the question: Does a Hausdorff (regular or Tychonoff) paratopological group of countable π-character have a G_δ-diagonal of infinite rank? In Theorem 2.8, we answer this question in the positive. Also, we show that if G is a Hausdorff ω-balanced paratopological group with countable Hausdorff number and countable pseudocharacter, then G has a G_δ-diagonal of infinite rank (see Proposition 2.12). This fact generalizes Arhangel'skii–Bella's result mentioned previously, since every Hausdorff first countable paratopological group is ω-balanced, has countable Hausdorff number and countable pseudocharacter.

We show that every Hausdorff Lindelöf paratopological group with countable pseudocharacter has a G_δ-diagonal of infinite rank (see Corollary 2.15). The same conclusion remains valid for regular precompact paratopological groups with countable pseudocharacter (see Corollary 2.16). However, in Example 2.18, we present a Hausdorff commutative precompact paratopological group of countable pseudocharacter which has no G_δ-diagonal of rank 3.

2. Paratopological groups with a G_δ-diagonal of infinite rank

Given a semitopological group G with identity e, denote by $\mathcal{N}(e)$, the family of open neighborhoods of e in G.

The following lemmas play an important role in the proof of Theorem 2.4.

Lemma 2.1. *If G is a Hausdorff semitopological group with countable π-character, then G has countable pseudocharacter.*

Proof. Let $\gamma = \{U_n : n \in \omega\}$ be a local π-base at the identity e in G. Put $W_n = U_n U_n^{-1}$. We claim that $\bigcap_{n\in\omega} W_n = \{e\}$. Indeed, take $x \in G\setminus\{e\}$. Since G is Hausdorff, there exists $V \in \mathcal{N}(e)$ such that $xV \cap V = \emptyset$. Since γ is a local π-base at e, we can find $n \in \omega$ such that $U_n \subseteq V$. We conclude that $xU_n \cap U_n = \emptyset$, i.e., $x \notin U_n U_n^{-1} = W_n$. □

Lemma 2.2. *Suppose that G is a semitopological group, A is a subset of G and γ is local π-base at the identity e in G. Then*

$$\bigcap_{V\in\gamma} AV^{-1} \subseteq \overline{A}.$$

Proof. We know that $\bigcap_{U\in\mathcal{N}(e)} AU^{-1} = \overline{A}$. If the intersection $\bigcap_{V\in\gamma} AV^{-1}$ is empty, we are done. Otherwise, take $x \in \bigcap_{V\in\gamma} AV^{-1}$. Let us show that $x \in \bigcap_{U\in\mathcal{N}(e)} AU^{-1}$. Take $U \in \mathcal{N}(e)$. Since γ is a local π-base at e, there exists $V \in \gamma$ satisfying $V \subseteq U$. Hence $x \in AV^{-1} \subseteq AU^{-1}$. We have thus proved that

$$\bigcap_{V\in\gamma} AV^{-1} \subseteq \bigcap_{U\in\mathcal{N}(e)} AU^{-1} = \overline{A}. \qquad \square$$

Given a collection $\mathcal{U}$ of subsets of a space X, the star of $\mathcal{U}$ respect to a subset $A \subseteq X$ is the set $st(\mathcal{U}, A) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. When $A = \{x\}$, we simply write $st(\mathcal{U}, x)$. We put $st^1(\mathcal{U}, x) = st(\mathcal{U}, x)$ and recursively define $st^{n+1}(\mathcal{U}, x) = st(\mathcal{U}, st^n(\mathcal{U}, x))$.

We say that a space X has a G_δ-diagonal of rank $k \in \mathbb{N}$ if there exists a countable collection $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that $\bigcap\{st^k(\mathcal{U}_n, x) : n \in \omega\} = \{x\}$, for each $x \in X$. In this case, we say that $\{\mathcal{U}_n : n \in \omega\}$ is a *diagonal sequence* of rank k on the space X. If a space X has a G_δ-diagonal of any possible rank, then we say that X has a G_δ-diagonal of infinite rank.