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Dual properties of monotonically normal spaces and generalized trees ☆

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ABSTRACT

In the first part of this note, we show that every monotonically normal space is dually scattered of rank ≤ 2.

In the second part of this note, we introduce a notion of a generalized tree with the Sorgenfrey topology (a generalized Sorgenfrey topology). A partially ordered set X is said to be a *generalized tree* if $(\leftarrow, x) = \{y \in X : y < x\}$ is a linearly ordered set for each $x \in X$. We say that a generalized tree X has *the Sorgenfrey topology* (a *generalized Sorgenfrey topology*) if each $x \in X$ with $(\leftarrow, x) = \emptyset$ is an isolated point, and for each $x \in X$ with $(\leftarrow, x) \neq \emptyset$, $\{(y, x] : y \in (\leftarrow, x)\}$ is a neighborhood base at x (or x is an isolated point).

We get the following conclusions. A topological space X is monotonically normal and homeomorphic to some generalized tree with some generalized Sorgenfrey topology if and only if X is a topological sum such that each factor is homeomorphic to a linearly ordered set with a generalized Sorgenfrey topology.

For a topological space X, the following are equivalent:

(a) X is monotonically normal and homeomorphic to some generalized tree with the Sorgenfrey topology.
(b) X is a topological sum such that each factor is homeomorphic to a linearly ordered set with the Sorgenfrey topology.
(c) The condition (c1) or (c2) below holds.
  (c1) X is homeomorphic to some linearly ordered set with the Sorgenfrey topology.
  (c2) X is a topological sum having at least two, but finitely many factors, and each factor is homeomorphic to an ordinal of uncountable cofinality.

We also get some conclusions on subspaces of ordinals which relate to a generalized tree with the Sorgenfrey topology.

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1. Introduction

A *neighborhood assignment* for a space X is a function ϕ from X to the topology of the space X such that $x \in \phi(x)$ for any $x \in X$ [15]. A space X is said to be *dually discrete* if for any neighborhood assignment ϕ for X there exists a discrete subspace D of X such that $X = \bigcup\{\phi(d) : d \in D\}$ [16]. We denote $\bigcup\{\phi(d) : d \in D\}$ by $\phi(D)$. A set D is called a *kernel* of a neighborhood assignment ϕ for X if $X = \phi(D)$. Some properties of dually discrete spaces can be found in [1] and [16].

In [4], it was proved that every subspace of an ordinal is dually discrete. Recall that a space X is a *generalized ordered space* (abbreviated *GO-space*) if it is embeddable in a linearly ordered topological space. In 2008, Peng proved that any GO-space is dually discrete [11]. We know that every GO-space is monotonically normal. The following problem appears in [4]: Must every monotonically normal space be dually discrete? In [17], it was proved that every neighborhood assignment for a monotonically normal space has a kernel which is homeomorphic to some subspace of an ordinal and showed that any monotone neighborhood assignment for a monotonically normal space has a discrete kernel.

In the first part of this note, we show that every monotonically normal space is dually scattered of rank ≤ 2. By this result, there seems more reasons to believe that every monotonically normal space must be dually discrete. In [6] and [13], it was also proved that the product of two ordinals is hereditarily dually discrete. In getting the conclusion that every subspace of the product of two ordinals is dually discrete [13], it was firstly showed that every subspace of the product of two ordinals is dually scattered of rank ≤ 2 [12].

In the second part of this note, we introduce a notion of a generalized tree with the Sorgenfrey topology (a generalized Sorgenfrey topology). A partially ordered set X is said to be a *generalized tree* if $(\leftarrow, x) = \{y \in X : y < x\}$ is a linearly ordered set for each $x \in X$. We say that a generalized tree X has *the Sorgenfrey topology* (a *generalized Sorgenfrey topology*) if each $x \in X$ with $(\leftarrow, x) = \emptyset$ is an isolated point, and for each $x \in X$ with $(\leftarrow, x) \neq \emptyset$, $\{(y, x] : y \in (\leftarrow, x)\}$ is a neighborhood base at x (or x is an isolated point). We get the following conclusions.

A topological space X is monotonically normal and homeomorphic to some generalized tree with some generalized Sorgenfrey topology if and only if X is a topological sum such that each factor is homeomorphic to a linearly ordered set with a generalized Sorgenfrey topology.

For a topological space X, the following are equivalent:

(a) X is monotonically normal and homeomorphic to some generalized tree with the Sorgenfrey topology.
(b) X is a topological sum such that each factor is homeomorphic to a linearly ordered set with the Sorgenfrey topology.
(c) The condition (c1) or (c2) below holds.
  (c1) X is homeomorphic to some linearly ordered set with the Sorgenfrey topology.
  (c2) X is a topological sum having at least two, but finitely many factors, and each factor is homeomorphic to an ordinal of uncountable cofinality.

We also get some conclusions on subspaces of ordinals which relate to a generalized tree with the Sorgenfrey topology.

All spaces in this note are assumed to be T_1-spaces. The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [5]. Let $\alpha, \beta, \gamma, \Lambda$ denote ordinals in this note.

2. Dual properties of monotonically normal spaces

Let's recall that a space X is *scattered* if every non-empty subspace of X has an isolated point. Let $X^* = \{x : x \in X$ and x is not an isolated point of $X\}$. If $X^* = \emptyset$ or X^* is a discrete subspace of X, then