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Primitive stable closed hyperbolic 3-manifolds

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ABSTRACT

We show that closed 3-manifolds with high Heegaard distance and bounded subsurface Heegaard distance are primitive stable when they are regarded as representations from the free group corresponding to the handlebody. This implies that any point on the boundary of Schottky space can be approximated by primitive stable representations corresponding to closed hyperbolic 3-manifolds.

1. Introduction

We consider a non-abelian free group F and the character variety $\chi(F)$ of the representations of F into $\mathrm{PSL}_2\,\mathbb{C}$. The outer automorphism group $\mathrm{Out}\,F$ of F acts on $\chi(F)$, and through work of Lubotzky [9], people took interest in its dynamics. It was well known that $\mathrm{Out}\,F$ acts properly discontinuously on the Schottky space, and it had been asked whether the Schottky space is exactly the domain of discontinuity for $\mathrm{Out}\,F$. Minsky [13] recently introduced a notion of primitive stability for $\mathrm{PSL}_2\,\mathbb{C}$ representations of free groups. He showed that the primitive stable representations form an open subset of the character variety which is bigger than the Schottky space and on which the outer automorphism group acts properly discontinuously.

Let us briefly recall the definition of primitive stable representations (see [13] for a complete definition). In a free group, an element is called *primitive* if it can be a member of a free generating set. A representation

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$\rho : F \to \mathrm{PSL}_2\,\mathbb{C}$ of a free group is *primitive stable* if it has the property that any ρ-equivariant map from a Cayley graph of F to $\mathbb{H}^3$ takes the geodesics defined by primitive elements to uniform quasi-geodesics.

In [14], Minsky and Moriah constructed lattices which are the images of primitive stable representations and asked whether any lattice is the image of such a primitive stable representation. In this note, we shall give further examples of lattices which are images of primitive stable representations by giving some sufficient conditions for primitive stability for closed hyperbolic 3-manifolds. To be more concrete, we shall show that under some conditions (large Hempel distance and bounded combinatorics), a Heegaard splitting gives two primitive stable representations of the free groups corresponding to the two handlebodies constituting the splitting. As an application of our main result, we shall also show that every boundary point of the Schottky space is a limit of primitive stable representations corresponding to closed hyperbolic 3-manifolds.

2. Statement of main theorem

Throughout this paper we assume all manifolds to be closed and orientable, and all Heegaard surfaces to have genus at least 2. To keep this note short we shall omit most of the well-known definitions. The reader should refer to [4] and [10] for the definitions of the curve complex and the curve graph, to [11] for the definition of subsurface projections, and to [5] for the definition of the Hempel (or Heegaard) distance (called the distance of the splitting in [5]).

The next definition, due to Namazi ([15]), is not (yet) so well known.

Definition 2.1. Let $M = H^1 \cup_\Sigma H^2$ be a Heegaard splitting. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the set of isotopy classes of meridians in H^1 and H^2 respectively, regarded as subsets in the curve graph of Σ. We say that the Heegaard splitting has *R-bounded combinatorics* when there is a pants decomposition P_j whose components are contained in $\mathcal{D}_j$ for $j = 1, 2$ such that for any essential subsurface Y of Σ the distance between the projections of P_1 and P_2 on Y is bounded by R.

The main theorem of this note is the following.

Theorem 2.2. *For any R, there exist K depending only on R and the genus g as follows. For any 3-manifold admitting a genus-g Heegaard splitting $M = H^1 \cup_\Sigma H^2$ whose Hempel distance is greater than K and which has R-bounded combinatorics, the manifold M is hyperbolic and the representation $\iota_* : \pi_1(H^j) \to \pi_1(M) \subset \mathrm{PSL}_2\,\mathbb{C}$ is primitive stable for $j = 1, 2$.*

Our proof of this theorem relies on the result of Namazi [15] on model manifolds for Heegaard splittings with R-bounded combinatorics and on the characterisation of primitive stable discrete and faithful representations given in [6]. As long as $K \geq 3$, the manifold M is hyperbolic by the Geometrisation Theorem, but Namazi gave an alternative proof of the hyperbolicity of M for K large enough as in our statement, which does not use the full Geometrisation Theorem. See [15].

3. Proof of the main theorem

The proof of the main theorem relies on the work of Namazi [15] on geometric models associated with Heegaard splittings and on the following result of [6].

Theorem 3.1 *(Jeon–Kim–Ohshika–Lecuire). Every discrete, faithful, and purely loxodromic representation of F is primitive stable.*

We shall prove the contrapositive of our statement.