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An infinite game with topological consequences

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ABSTRACT

We introduce a two player infinite game for which a winning strategy is preserved by Σ-products and implies the existence of a winning strategy in Gruenhage's W-space game. A space exhibiting a winning strategy is collectionwise normal and countably paracompact. Consequently, a Σ-product of spaces is collectionwise normal if each space possesses a winning strategy. This is a generalization of a well-known result on Σ-products of metrizable spaces. Finally we show certain uniform box products have a winning strategy.

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1. Introduction

Infinite games can be used to define certain types of topological spaces [5,13]. We introduce an infinite game played in a uniform space for which we show a winning strategy implies certain topological properties. Every metric space exhibits a winning strategy, but not every space with a winning strategy is metrizable. Furthermore every space with a winning strategy is a W-space as well as collectionwise normal and countably paracompact. However, there exist paracompact spaces with no winning strategy, and spaces with a winning strategy which are not paracompact. There is a natural weakening of the game winning criteria, and a space with a winning strategy in this weak sense is countably metacompact and collectionwise Hausdorff.

Σ-products of spaces, originally investigated by H.H. Corson [3], are not necessarily normal even if the factor spaces are compact [2]. In Section 6 we show a winning strategy is preserved by closed subspaces and Σ-products. As a consequence, a Σ-product of compact spaces is normal if each factor space has a winning strategy. It is known that Σ-products of metrizable spaces are collectionwise normal and countably paracompact [12]. In Sections 7 and 9 we obtain a generalization of this result.

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As a further application, in Section 10 we show certain uniform box products have a winning strategy, and deduce the previously unknown result that these spaces are collectionwise normal and countably paracompact.

2. Preliminaries

All spaces are assumed Hausdorff. The following examples will be used throughout this paper. The *Sorgenfrey line* S is the space of real numbers where intervals of the form $[a, b)$ are a basis, and the *Sorgenfrey plane* is $S \times S$ with the product topology. A *Fort space* is the one point compactification of a discrete space. If W is a discrete space we will denote the one point compactification X of W by $X = W \cup \{\infty\}$. If W is uncountable, we say X is an uncountable Fort space. We denote the closure of a set A by $cl(A)$.

We include some background on uniform spaces and refer the reader to [4] or [7] for further details.

If X is a set and if $D \subseteq X \times X$, then $D^{-1} = \{(y, x) : (x, y) \in D\}$ and $D \circ D = \{(x, z) : \exists y$ such that $(x, y) \in D$ and $(y, z) \in D\}$. If $D = D^{-1}$ we say D is *symmetric*. The *diagonal* $\{(x, x) : x \in X\}$ will be denoted by $\triangle$. For a subset A of X, let $D[A] = \{y : (x, y) \in D$ for some $x \in A\}$ and for $x \in X$ abbreviate $D[\{x\}]$ by $D[x]$. We will often write $2D$ as shorthand for $D \circ D$ and $4D$ for $D \circ D \circ D \circ D$.

Definition 1. A *uniformity* $\mathbb{D}$ on a set X is a collection of subsets, known as *entourages*, of $X \times X$ such that

1. for all $D \in \mathbb{D}$, $\triangle \subseteq D$
2. if $D \in \mathbb{D}$ then $D^{-1} \in \mathbb{D}$
3. if $D \in \mathbb{D}$ then $E \circ E \subseteq D$ for some $E \in \mathbb{D}$
4. if D and E are in $\mathbb{D}$ then $D \cap E \in \mathbb{D}$
5. if $D \in \mathbb{D}$ and $D \subseteq E$ then $E \in \mathbb{D}$

A *uniform space* is a pair $(X, \mathbb{D})$, where X is a set and $\mathbb{D}$ is a uniformity on X.

Every uniformity $\mathbb{D}$ on a set X defines a topology on X, by declaring $G \subseteq X$ to be open provided for every $x \in G$ there is $D \in \mathbb{D}$ such that $D[x] \subseteq G$. This topology on X is the *uniform topology* induced by $\mathbb{D}$. A subfamily $\mathbb{E}$ of $\mathbb{D}$ is a *base* for $\mathbb{D}$ if every member of $\mathbb{D}$ contains a member of $\mathbb{E}$. For convenience we will assume every uniformity base contains $X \times X$. We sometimes refer to a uniform space $(X, \mathbb{D})$ by the notation $(X, \mathbb{E})$, where $\mathbb{E}$ is a base for a uniformity $\mathbb{D}$ on X. An entourage $D \in \mathbb{D}$ is *open* if D is open in $X \times X$ with the product topology. The family of all open, symmetric members of $\mathbb{D}$ is a base for $\mathbb{D}$ [7].

If X is already a topological space, then a uniformity $\mathbb{D}$ on X is *compatible* with the topology on X provided the uniform topology generated by $\mathbb{D}$ coincides with the topology on X. A topological space X is *uniformizable* if there is a uniformity $\mathbb{D}$ on X compatible with the topology on X. The uniformizable spaces are precisely the completely regular spaces. A uniform space is metrizable if and only if $\mathbb{D}$ has a countable base [4].

The following facts, which will be used later, are proven here for completeness.

Lemma 1. *Suppose $(X, \mathbb{D})$ is a uniform space, $D \in \mathbb{D}$ is symmetric and $x \in X$. If $A \subseteq D[x]$ and if $y \in cl(A)$, then $cl(A) \subseteq 4D[y]$.*

Proof. Suppose $A \subseteq D[x]$ and $y \in cl(A)$. Then $cl(D[x]) \subseteq 2D[x]$ and so $(x, y) \in 2D$, whence $(y, x) \in 2D$ since D is symmetric. If $z \in cl(A)$, then $(x, z) \in 2D$. Then $(y, z) \in 4D$ and so $z \in 4D[y]$, therefore $cl(A) \subseteq 4D[y]$. □

Definition 2. An open cover $\mathscr{U}$ of a space X is a *normal cover* provided there is a sequence $\mathscr{V}_0, \mathscr{V}_1, \ldots$ of open covers of X such that $\mathscr{V}_0 = \mathscr{U}$ and $\mathscr{V}_{n+1}$ star refines $\mathscr{V}_n$.