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Algebraically determined semidirect products

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ABSTRACT

Let G be a Polish (i.e., complete separable metric topological) group. Define G to be an algebraically determined Polish group if given any Polish group L and an algebraic isomorphism $\varphi : L \mapsto G$, then φ is a topological isomorphism. The purpose of this paper is to prove a general theorem that gives useful sufficient conditions for a semidirect product of two Polish groups to be algebraically determined. This general theorem will provide a flowchart or recipe for proving that some special semidirect products are algebraically determined. For example, it may be used to prove that the natural semidirect product $\mathcal{H} \rtimes \mathcal{G}$, where $\mathcal{H}$ is the additive group of a separable Hilbert space and $\mathcal{G}$ is a Polish group of unitaries on $\mathcal{H}$ acting transitively on the unit sphere with $-I \in \mathcal{G}$, is algebraically determined. An example of such a $\mathcal{G}$ is the unitary group of a separable irreducible C^*-algebra with identity on $\mathcal{H}$. Not all nontrivial semidirect products of Polish groups are algebraically determined, for it is known that the Heisenberg group $\mathbb{H}_3(\mathbb{R})$ is a semidirect product of the form $\mathbb{R}^2 \rtimes_\theta \mathbb{R}^1$ and is not an algebraically determined Polish group.

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1. Introduction

Let G be a Polish (i.e., complete separable metric topological) group. Define G to be an algebraically determined Polish group if given any Polish group L and an algebraic isomorphism $\varphi : L \mapsto G$, then φ is a topological isomorphism. The study of algebraically determined Polish groups dates back to the work of E. Cartan, B.L. van der Waerden and H. Freudenthal, among many others. See the introduction to [1] for a condensed history of algebraically determined Polish groups and why they are of scientific interest.

The purpose of this paper is to prove a general result (Theorem 4) that gives a useful list of steps which suffice to prove that a semidirect product of two Polish groups is algebraically determined. This general theorem will provide a flowchart or recipe for proving that some special semidirect products are algebraically

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determined and is an abstraction and simplification of the techniques introduced in [1] and [4]. Theorem 4 is of a general character and the steps needed to verify its hypotheses in any particular instance can be quite difficult and require considerable ingenuity. That such individual ingenuity is required can be seen from a few examples. Observe that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is a trivial semidirect product that is not algebraically determined. Furthermore, not all nontrivial semidirect products of Polish groups are algebraically determined, for it is known that the Heisenberg group $\mathbb{H}_3(\mathbb{R})$, consisting of all 3×3 upper triangular matrices with 1's along the diagonal, is a semidirect product of the form $\mathbb{R}^2 \rtimes_\theta \mathbb{R}^1$ and is not an algebraically determined Polish group [4].

Kechris and Rosendal [6] introduced the notion of Polish groups with ample generics. This is an important but very special class of Polish groups since any homomorphism of such a Polish group into a separable topological group is automatically continuous. Obviously any Polish group with ample generics is algebraically determined. The converse is false since it is easy to see that any connected Lie group cannot have ample generics even though there are many examples of Lie groups that are algebraically determined. For example, the known result that the real $ax + b$ group, a very elementary semidirect product, is algebraically determined follows easily from Theorem 4. However, the real $ax + b$ group cannot have ample generics since its natural injection into the complex $ax + b$ group followed by a discontinuous automorphism of $\mathbb{C}$ is a discontinuous homomorphism into a separable group. Therefore Theorem 4 applies to a much wider class of Polish groups than merely those with ample generics.

Some new results are easy consequences of Theorem 4. For example, it may be used to prove that the natural semidirect product $\mathcal{H} \rtimes \mathcal{G}$, where $\mathcal{H}$ is the additive group of a separable Hilbert space and $\mathcal{G}$ is a Polish group of unitaries on $\mathcal{H}$ acting transitively on the unit sphere with $-I \in \mathcal{G}$, is algebraically determined. An example of such a $\mathcal{G}$ is the unitary group $\mathcal{U}(\mathcal{A})$ of a separable irreducible C^*-algebra $\mathcal{A}$ with identity acting on $\mathcal{H}$ is algebraically determined.

The mathematical tools used in this paper are descriptive set theory methods. Basic references can be found in [2,5,7] and [8].

2. The mathematical tools

If X is a topological space, let $\mathcal{B}(X)$, the Borel subsets of X, be the σ-algebra generated by the open subsets of X and let $\mathcal{BP}(X)$, the subsets of X with the Baire property, be the σ-algebra generated by $\mathcal{B}(X)$ and the meager subsets of X. If X is a Polish space, then $\mathcal{BP}(X)$ contains the analytic subsets of X. The following proposition, a slight paraphrase of Theorem 1.2.6 in [2], is the basic general principal used to establish the results discussed in this paper. It is a consequence of the Banach–Kuratowski–Pettis theorem (Theorem 9.09, [5]).

Proposition 1. *Let H and G be two Polish groups and let $\varphi : H \mapsto G$ be an algebraic homomorphism that is $\mathcal{BP}(H)$-measurable. Then φ is continuous. Furthermore, φ is open if $\varphi(H)$ is nonmeager. In particular, if φ is an algebraic isomorphism, then φ is a topological isomorphism.*

The following proposition will prove to be most useful.

Proposition 2. *(Proposition 5, [1]) Let G be a Polish topological group, $A \subset G$ an analytic subset and $H \subset G$ an analytic subgroup such that A intersects each H-coset in exactly one point and $G = AH$. Then H is closed in G.*

3. The main result

Recall that if K and Q are groups and $\theta : Q \mapsto Aut(K)$, $q \mapsto \theta_q$, then $K \rtimes_\theta Q$, the semidirect product of K and Q, is the group whose underlying set is $K \times Q$ and whose multiplication is $(k_1, q_1)(k_2, q_2) =$