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Degree of homogeneity on cones

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ABSTRACT

The degree of homogeneity of a space X is the number of orbits for the action of the group of homeomorphisms of X onto itself. In this paper we determine the degree of homogeneity of the cone of a space X in terms of that of X, in the case in which X is either a local dendrite or a Hausdorff space with no arcs.

1. Introduction

Let $\mathcal{H}(X)$ denote the group of homeomorphisms of a space X onto itself. An *orbit of* X is the action of $\mathcal{H}(X)$ at a point x_0 of X, namely $\mathcal{O}_X(x_0) = \{h(x_0) : h \in \mathcal{H}(X)\}$. Given a positive integer n, a space is said to be $\frac{1}{n}$*-homogeneous* provided that X has exactly n orbits, in which case we say that the *degree of homogeneity of* X is n. Since 2006 there has been increasing interest in the study of $\frac{1}{2}$-homogeneity, in fact, several papers have been written on the subject: [1,2,6,10–13,16–24]. Higher degrees of homogeneity appear to be studied only in [8,26].

In [26] the degree of homogeneity of the suspension of a local dendrite X in terms of that of X is fully determined. In the present paper we determine the orbits of the cone of X—in terms of those of X—for

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some spaces X, namely local dendrites and Hausdorff spaces with no arcs; in particular we show that no local dendrite has $\frac{1}{3}$-homogeneous cone. We also analyze how some subsets of $Cone(X)$ behave under homeomorphisms of $Cone(X)$ onto itself.

This paper is organized in eight sections. After Notation and Terminology, in Sections 3 and 4 we present some basic results and we introduce some important sets that will be used throughout the paper. In Section 5 we study part of the behavior of the orbits of $Cone(X)$ when X is a finite graph or dendrite and we use such results to obtain generalizations for local dendrites in Section 6.

In Section 7 we present the main result of this article, namely we give a formula to calculate the degree of homogeneity of a cone of a local dendrite. We give some corollaries as well. Finally, in Section 8 we determine the degree of homogeneity of a cone of a Hausdorff space with no arcs X in terms of that of X. We end the paper with some open questions.

2. Notation and terminology

In this section we present general notation and we also introduce terminology that we use frequently. For notation and terminology not given here or in Section 1 see [15].

The symbol $\mathbb{N}$ denotes the set of positive integers and $A \times B$ denotes the Cartesian product of A and B; also $|A|$ denotes the cardinality of a set A and $\operatorname{diam}(A)$ denotes its diameter.

Let X be a topological space and $A \subset X$. The symbol $\overline{A}^X$ denotes the closure of A in X; $\operatorname{int}_X(A)$ denotes the topological interior of A in X; $\operatorname{bd}_X(A)$ denotes the topological boundary of A in X and A' denotes the set of accumulation points of A in X.

For a manifold M, the symbols iM and ∂M will denote the manifold interior and the manifold boundary of M, respectively.

Recall that for a topological space X, the cone of X, $Cone(X)$, is the quotient space that is obtained by identifying all the points $(x,1)$ in $X \times [0,1]$ to a single point (see [15, 3.15, p. 41]). We denote the vertex of $Cone(X)$ by v_X. We often assume that $X \times [0,1)$ is a subspace of $Cone(X)$; with this in mind, we write points in $Cone(X)$ that are not the vertex as ordered pairs (x,t). If $A \subset X$, then we consider $Cone(A)$ as a subset of $Cone(X)$ with the same vertex, v_X, as $Cone(X)$. Moreover, we use the symbol π to denote the natural projection of $Cone(X) \setminus \{v_X\}$ onto X, that is $\pi(x,t) = x$, for all $(x,t) \in Cone(X) \setminus \{v_X\}$.

Further, for a topological space X the suspension of X, $Sus(X)$, is the quotient space that is obtained from $X \times [-1,1]$ by identifying $X \times \{1\}$ to a single point v_X^1 and $X \times \{-1\}$ to another point v_X^{-1} (see [15, 3.16, p. 42]).

A *continuum* is a nonempty, compact and connected metric space; it is well known that if X is a continuum, so is $Cone(X)$ [15, p. 42].

An *arc* is a space homeomorphic to the closed interval $[0,1]$. An arc A with end points p and q in a space X is a *free arc in* X provided that $A \setminus \{p,q\}$ is open in X. By a *maximal free arc* we mean a free arc that is not properly contained in any free arc.

A *simple closed curve* is a space homeomorphic to the unit circle S^1. By a *loop* in a space X we mean a simple closed curve C in X such that $\operatorname{bd}_X(C) = \{v\}$ for some $v \in X$.

Define the following families of subsets of a continuum X:

$$\mathcal{L}_X = \{J: J \text{ is a maximal free arc in } X\} \quad \text{and}$$
$$\mathcal{S}_X = \{L: L \text{ is a loop in } X\}. \tag{1}$$

We will often consider the end points of an element $A \in \mathcal{L}_X \cup \mathcal{S}_X$; note that when we say that p and q are the end points of A and $A \in \mathcal{S}_X$, we understand that $\{p\} = \{q\} = \operatorname{bd}_X(A)$.

By a *finite graph* we mean a continuum that can be expressed as the union of finitely many arcs, any two of which intersect in at most one or both of their end points [15, 9.1, p. 140].