Topology and its Applications 175 (2014) 65–71

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Note on countable closed discrete sets in products of natural numbers

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ARTICLE INFO

Article history:
Received 21 May 2013
Received in revised form 11 February 2014
Accepted 16 July 2014
Available online 26 July 2014

MSC:
primary 54B10, 54D15, 54C45
secondary 54D60

Keywords:
Products of natural numbers
Non-normality
Countable closed discrete sets
C- and $C^\star$-embeddings

ABSTRACT

We provide several examples concerning extensions of real-valued functions on countable closed discrete subsets of products $\mathbb{N}^{\omega_1}$ or $\mathbb{N}^{2^\omega}$ of natural numbers over the whole products.

1. Introduction

Our terminology follows [9]. A subset A of a topological space X is C ($C^\star$)-embedded in X if any continuous real-valued (and bounded) function on A extends continuously over X. We shall denote by $\mathbb{N}^\kappa$ the product of κ copies of natural numbers $\mathbb{N}$.

For a countable closed discrete set A in $\mathbb{N}^\kappa$, C ($C^\star$)-embedding means that each function from A to $\mathbb{N}$ (to $\{0,1\}$) can be extended over $\mathbb{N}^\kappa$ to a continuous function with the same range, and if some injection of A into $\mathbb{N}$ has such an extension, then A is C-embedded in $\mathbb{N}^\kappa$, cf. [9], 3L and [11], Section 1.

In a recent interesting paper, Keith M. Fox [8] constructed a variety of countable closed discrete not $C^\star$-embedded sets in $\mathbb{N}^{\omega_1}$.

We shall use a different approach, based on [19,15,17,3] to the following effect:

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http://dx.doi.org/10.1016/j.topol.2014.07.005

Example 1.1. There is a countable closed discrete set A in $\mathbb{N}^{\omega_1}$ and an uncountable almost disjoint collection $\mathcal{A}$ of infinite subsets of A such that

(i) for each countable $\mathcal{C} \subset \mathcal{A}$ there is a C-embedded in $\mathbb{N}^{\omega_1}$ subset of A which almost contains each member of $\mathcal{C}$,
(ii) there are no $C^\star$-embedded in $\mathbb{N}^{\omega_1}$ subsets of A which almost contain uncountably many members of $\mathcal{A}$.

If we assume that $\omega_1 < \mathfrak{r}$ (cf. [2]), one can assert in (ii) that no subset of A which has infinite intersection with uncountably many members of $\mathcal{A}$, is $C^\star$-embedded in $\mathbb{N}^{\omega_1}$, cf. Section 4.1.

We shall also show that in $\mathbb{N}^{2^\omega}$ there is a countable closed discrete set A which can be split into two sets such that for no $\kappa < 2^\omega$ these sets can be enlarged to unions of intersections of κ many zero-sets with disjoint closures, and there is a bijection of A onto the set of rationals in the open interval $(0,1)$ taking $C^\star$-embedded in $\mathbb{N}^{2^\omega}$ sets onto sets with countable closures in $[0,1]$ and vice versa, cf. Section 3.

The particular set A that we shall construct has the property that $C^\star$-embedded in $\mathbb{N}^{2^\omega}$ subsets of A are also C-embedded in $\mathbb{N}^{2^\omega}$. Let us recall, however, that this is not always the case, cf. Section 4.3.

2. Construction of Example 1.1

Let us recall a theorem of Mycielski [15], cf. [10], A1.6:

(A) *There is a closed discrete set $M = \{a_\alpha : \alpha < \omega_1\}$ in $\mathbb{N}^{\omega_1}$ such that for any $\alpha < \omega_1$ the projection of $M_\alpha = \{a_\beta : \beta < \alpha\}$ onto some coordinate is injective.*

By a theorem of Rothberger [19], Theorem 3 (cf. [13]), the Mycielski set is in the sequential closure of some countable subset of $\mathbb{N}^{\omega_1}$, i.e.,

(B) *there are points $s_n \in \mathbb{N}^{\omega_1}$, $n = 1, 2, \ldots$, such that for any $a_\alpha \in M$ there is a sequence $n_1 < n_2 < \ldots$ with $s_{n_i} \to a_\alpha$ in $\mathbb{N}^{\omega_1}$.*

We shall also use the following two observations, the first contained in a proof of Example 2 in [17] (cf. also Section 4.2) and the second one being Lemma 9.1 in [3]:

(C) *for each uncountable closed discrete set D in $\mathbb{N}^{\omega_1}$ there is a locally countable in $\mathbb{N}^{\omega_1}$ collection $\mathcal{E}$ of zero-sets such that $D \subset \bigcup \mathcal{E}$ and no zero-set in $\mathbb{N}^{\omega_1}$ contained in $\bigcup \mathcal{E}$ intersects D in an uncountable set,*
(D) *if $X \subset \mathbb{N}^{\omega_1}$ and $\mathbb{N}^{\omega_1}$ has an open cover by sets U such that $U \cap X$ admits a closed embedding in $\mathbb{N}^{\omega_1}$, then X embeds as a closed set in $\mathbb{N}^{\omega_1}$.*

It is worth recalling that if $X_\alpha \subset \mathbb{N}^{\omega_1}$, $\alpha < \omega_1$, embeds as a closed set in $\mathbb{N}^{\omega_1}$, so does $X = \bigcap_\alpha X_\alpha$, as X can be identified with the diagonal of the product $\prod_\alpha X_\alpha$. Since for each zero set Z in $\mathbb{N}^{\omega_1}$, $\mathbb{N}^{\omega_1} \setminus Z$ embeds as a closed set in $\mathbb{N}^{\omega_1}$ ($\mathbb{N}^{\omega_1} \setminus Z$ is homeomorphic with $\bigcup_n U_n \times \{n\}$, where U_n are closed-and-open in $\mathbb{N}^{\omega_1}$, pairwise disjoint and cover $\mathbb{N}^{\omega_1} \setminus Z$), we infer that any complement of ω_1 zero-sets in $\mathbb{N}^{\omega_1}$ embeds as a closed set in $\mathbb{N}^{\omega_1}$, cf. [4], 8.18.

Let us check also the following simple fact:

(E) *if $L \subset \mathbb{N}^{\omega_1}$, $\mathcal{F}$ is a collection of countable closed discrete subsets of L which are C-embedded in L, $|\mathcal{F}| \le \omega_1$ and L admits a closed embedding in $\mathbb{N}^{\omega_1}$, then there is a closed embedding $g : L \to \mathbb{N}^{\omega_1}$ such that each $g(F)$, for $F \in \mathcal{F}$, is C-embedded in $\mathbb{N}^{\omega_1}$.*