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Coincidence points principle for mappings in partially ordered spaces

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ABSTRACT

The concept of covering (regularity) for mappings in partially ordered spaces is introduced. Sufficient conditions for the existence of coincidence points and minimal coincidence points of isotone and orderly covering mappings are obtained. These results generalize classical fixed point theorems for isotone mappings. Moreover, the known theorems on coincidence points of covering and Lipschitz mappings in metric spaces are deduced from the obtained results.

Given nonempty sets X, Y and mappings $\psi, \varphi : X \to Y$, a point $x \in X$ that satisfies the equality $\psi(x) = \varphi(x)$ is called a coincidence point of the mappings ψ and φ. In this paper we introduce sufficient conditions for the existence of coincidence points of two mappings.

In the case when X and Y are metric spaces, sufficiently complete results on the coincidence points existence were obtained in [1–3]. These results are based on the covering mapping concept (see [4,5]) which has sufficiently enlarged the collection of tools for dealing with operator equations.

The present paper is devoted to the problem of the coincidence points existence in the case when X and Y are partially ordered spaces. It turns out that the concept of covering can be modified and presented for mappings acting in partially ordered spaces. As a result, the statements on the existence of coincidence points for isotone and orderly covering mappings can be obtained. As special cases of such statements, there can be derived the classical theorems of G. Birkhoff, A. Tarski, B. Knaster, L.V. Kantorovich on fixed points of isotone mappings (see, for instance, [6, pp. 25–26], [7, p. 266]).

In the paper, it is also shown that the known theorems on coincidence points of covering and Lipschitz mappings in metric spaces (including classical fixed point theorems for contraction mappings) follow from the obtained here results on coincidence points of mappings in partially ordered sets. The development of

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these results to set-valued mappings is already obtained and will be published soon. The corresponding results were announced in [8,9] without proofs.

1. Preliminaries

Let $(X, \preceq)$ be a partially ordered set. Recall that a subset $S \subset X$ is called a **chain** if any two elements of S are comparable. A point $q \in X$ is called a **lower bound** of a set $Q \subset X$ if $q \preceq x$ for all $x \in Q$. A lower bound $\bar{q} \in X$ of Q is called the **infimum** of Q, and is denoted by $\inf Q$, if $q \preceq \bar{q}$ for any lower bound q of Q. A set $Q \subset X$ is called **bounded from below** if there exists a lower bound $q \in X$ of Q. A point $m \in Q$ is called **a minimal point** in the set Q if there is no point $q \in Q$ such that $q \prec m$. A point $m \in Q$ is called the **least point** in Q if $m \preceq q$ for all $q \in Q$. Any least point is a minimal point, the converse is not true.

We say that a subset $A \subset X$ is **orderly complete in** X, if for any chain $S \subset A$ there exists $\inf S \in X$ and $\inf S \in A$. If X is orderly complete with respect to X, then we will briefly call $(X, \preceq)$ **orderly complete**. It is obvious that X is orderly complete if and only if any chain $S \subset X$ has an infimum.

We say that a subset $A \subset X$ is **orderly σ-complete with respect to** X, if for any non-increasing sequence $\{x_n\} \subset A$ there exists $\inf\{x_n\}$ and $\inf\{x_n\} \in A$. If X is orderly σ-complete with respect to X, then we will briefly call $(X, \preceq)$ **orderly σ-complete**. It is obvious that $(X, \preceq)$ is orderly σ-complete if and only if any non-increasing sequence $\{x_n\} \subset X$ has an infimum.

These definitions imply that if $(X, \preceq)$ is orderly complete, then it is orderly σ-complete. The converse is not true. Consider the corresponding example.

Example 1. Let X be a system of all Lebesgue measurable subsets of $\mathbb{R}$ ordered by inclusion. It is a straightforward task to ensure that for any chain $S \subset X$, if $\inf S$ exists, then $\inf S = \bigcap_{A \in S} A$. Since the countable intersection of Lebesgue measurable sets is measurable, $(X, \subset)$ is orderly σ-complete. Let us show that $(X, \subset)$ is not orderly complete. Take an arbitrary nonmeasurable set $B \subset \mathbb{R}$. Let $\overline{X}$ be a system of all measurable sets A such that $B \subset A$. Obviously, $\overline{X} \neq \emptyset$ since $\mathbb{R} \in \overline{X}$. By the Hausdorff maximal principle, there exists a maximal chain $S \subset \overline{X}$. Let us prove that this chain does not have an infimum in X. Consider the contrary, i.e., there exists an infimum $\overline{A} \in X$ of S. Then $\overline{A}$ is measurable and $\overline{A} = \bigcap_{A \in S} A$. Moreover, $B \subset \overline{A}$ since $B \subset A$ for all $A \in S$, and $\overline{A} \neq B$ since B is nonmeasurable. Therefore, there exists a real number $a \in \overline{A} \setminus B$. Thus, $\overline{A} \setminus \{a\}$ is measurable, $B \subset (\overline{A} \setminus \{a\})$, $(\overline{A} \setminus \{a\}) \subset A$ for all $A \in S$. This contradicts the maximality of S in $\overline{X}$. Hence, the chain S has no infimum in X. So, $(X, \subset)$ is not orderly complete.

By analogy with the definitions given above, there can be introduced the concepts of **non-decreasing** sequence, upper bound, supremum, set bounded from above, maximal and greatest elements. Non-decreasing and non-increasing sequences are called **monotone**. A set is called **bounded** if it is bounded from above and from below. An example of a bounded set is the segment $[a, b]$ — the set of all $x \in X$ such that $a \preceq x \preceq b$, where $a, b \in X$ are given points.

2. Coincidence points

2.1. Basic concepts

Let $(X, \preceq)$, $(Y, \preceq)$ be partially ordered sets. A mapping $\varphi : X \to Y$ is called **isotone** if for any $x_1, x_2 \in X$, if $x_1 \preceq x_2$, then $\varphi(x_1) \preceq \varphi(x_2)$.

We say that a mapping $\varphi : X \to Y$ is **orderly continuous** if for any chain $S \subset X$ that has an infimum in X, infimum of $\varphi(S)$ exists in Y and

$$\inf\{\varphi(S)\} = \varphi(\inf\{S\}). \tag{1}$$