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F-Dugundji spaces, F-Milutin spaces and absolute F-valued retracts

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ABSTRACT

For every functional functor $F : \mathsf{Comp} \to \mathsf{Comp}$ in the category Comp of compact Hausdorff spaces we define the notions of F-Dugundji and F-Milutin spaces, generalizing the classical notions of Dugundji and Milutin spaces. We prove that the class of F-Dugundji spaces coincides with the class of absolute F-valued retracts. Next, we show that for a monomorphic continuous functor $F : \mathsf{Comp} \to \mathsf{Comp}$ admitting tensor products each Dugundji compact space is an absolute F-valued retract if and only if the doubleton $\{0, 1\}$ is an absolute F-valued retract if and only if some points $a \in F(\{0\}) \subset F(\{0, 1\})$ and $b \in F(\{1\}) \subset F(\{0, 1\})$ can be linked by a continuous path in $F(\{0, 1\})$. We prove that for the functor Lip_k of k-Lipschitz functionals with $k < 2$, each absolute Lip_k-valued retract is openly generated. On the other hand, the one-point compactification of any uncountable discrete space is not openly generated but is an absolute Lip_3-valued retract. More generally, each hereditarily paracompact scattered compact space X of finite scattered height $n = \mathrm{ht}(X)$ is an absolute Lip_k-valued retract for $k = 2^{n+2} - 1$.

1. Introduction

A classical Tietze–Urysohn Theorem [6, 2.1.8] says that each continuous function $f : X \to \mathbb{R}$ defined on a closed subset X of a normal topological space Y admits a continuous extension $\bar{f} : Y \to \mathbb{R}$. According to a classical theorem of Dugundji [5], for a closed subset X of a metrizable topological space Y and a locally convex linear topological space Z there is a linear operator $u : C(X, Z) \to C(Y, Z)$ extending each continuous function $f \in C(X, Z)$ to a function $\bar{f} \in C(Y, Z)$ with values in the closed convex hull $\overline{\mathrm{conv}}(f(X))$ of $f(X)$ in Z. Operators with this property will be called *regular*.

Here by $C(X, Z)$ we denote the linear space of all continuous maps from X to Z. The linear space $C(X, \mathbb{R})$ of real-valued continuous functions on a topological space X is usually is denoted by $C(X)$. If the space X is

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compact, then the linear space $C(X)$ carries the structure of a Banach lattice with respect to the sup-norm $\|f\| = \sup_{x\in X} |f(x)|$.

A natural temptation to unify Tietze–Urysohn and Dugundji Theorems fails as there are pairs (X, A) of compact Hausdorff (and hence normal topological) spaces $A \subset X$ admitting no regular linear extension operator $u : C(A) \to C(X)$. This circumstance led A. Pełczyński [14] to the idea of introducing the class of Dugundji compact spaces. Those are compact spaces X admitting for each embedding $X \hookrightarrow Y$ into a compact Hausdorff space Y a regular linear extension operator $u : C(X) \to C(Y)$.

The systematic study of the class of Dugundji compact spaces was started by A. Pełczyński in [14]. Soon, it was realized that Dugundji compact spaces can be characterized as absolute P-valued retracts for the functor $P : \mathsf{Comp} \to \mathsf{Comp}$ of probability measures in the category Comp of compact Hausdorff spaces and their continuous maps. Let us recall that for a compact Hausdorff space X its space of probability measures PX is the subspace of the Tychonoff power $\mathbb{R}^{C(X)}$ consisting of all regular linear functionals $\mu : C(X) \to \mathbb{R}$ (the regularity of μ means that $\mu(f) \subset \overline{\mathrm{conv}}(f(X))$). Each point $x \in X$ can be identified with the Dirac measure $\delta_x : C(X) \to \mathbb{R}$, assigning to each function $f \in C(X)$ its value $f(x)$ at x. The assignment $x \mapsto \delta_x$ defines a canonical embedding $\delta : X \to PX$ of X into its space of probability measures.

A compact Hausdorff space X is called an *absolute P-valued retract* if for each embedding $X \subset Y$ into a compact Hausdorff space Y there is a continuous map $f : Y \to PX$ extending the canonical embedding $\delta : X \to PX$, see [8] for more details.

A breakthrough in understanding the structure of Dugundji compacta was made by R. Haydon [10] who proved that the class of Dugundji compacta coincides with the class $\mathsf{AE}(0)$ of compact absolute extensors in dimension zero. We say that a topological space X is an *absolute extensor in dimension n* if each continuous map $f : B \to X$ defined on a closed subspace B of a compact Hausdorff space A of dimension $\dim(A) \le n$ admits a continuous extension $\bar{f} : A \to X$. By $\mathsf{AE}(n)$ we shall denote the class of compact absolute extensors in dimension n.

Theorem 1.1 *(Haydon). For a compact Hausdorff space X the following conditions are equivalent:*

(1) *X is a Dugundji compact space;*
(2) *X is an absolute P-valued retract;*
(3) *X is an absolute extensor in dimension 0.*

The implication (3) ⇒ (1) of this theorem is usually proved with help of Milutin compact spaces, see [9]. Let us recall [9] that a compact Hausdorff space X is *Milutin* if there is a continuous surjective map $f : K \to X$ from a Cantor cube $K = \{0, 1\}^\kappa$, admitting a regular averaging operator $u : C(K) \to C(X)$, i.e., a regular linear operator such that $u(\varphi \circ f) = \varphi$ for any $\varphi \in C(X)$. In [13] Milutin proved that the unit interval $\mathbb{I} = [0, 1]$ is Milutin and derived from this fact that each Dugundji compact space is Milutin. The converse is not true as shown by the example of the hyperspace $\exp_2(\{0, 1\}^{\aleph_2})$ which is Milutin but not Dugundji, see [9, 6.7].

Theorem 1.1 shows that Dugundji compact spaces are tightly connected with the functor of probability measures P (this was observed and widely exploited by Ščepin in [22]). The relation of the class of Dugundji spaces to some other functors was studied by Alkins and Valov [1,25].

In this paper for any functional functor $F : \mathsf{Comp} \to \mathsf{Comp}$ we define the notions of F-Dugundji and F-Milutin compact spaces and will characterize these spaces in terms of extension and averaging operators between the spaces of continuous functions thus generalizing Theorem 1.1 to other functors. In particular, we shall prove that the class of F-Dugundji compact spaces coincides with the class of absolute F-valued retracts. In Sections 3 and 4 for certain (concrete functional) functors we shall study the class of absolute F-valued retracts and its relation to the classes of Dugundji compact spaces and of openly generated compacta.