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Approximation theorems in metric spaces and functionals strictly subordinated to convergent series

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A R T I C L E   I N F O

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Dedicated to the memory of V.V. Fedorchuk, an outstanding mathematician and a wonderful person

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A B S T R A C T

Some new results are presented concerning the search and approximation of solutions of equations in metric spaces using functionals strictly subordinated to a convergent series and functionals compatible with a convergent series. These classes of functionals were earlier introduced by the author.

The problems of search and approximation of solutions of equations in metric spaces are considered.

The fixed point search and approximation problems of a given self-mapping of a metric space go back to classical Banach contraction principle and its diverse generalizations. Many classical results in this area are collected in the remarkable book [1] (see also [3,4]). Let us list some of them. For example, the fixed point theorem in a complete metric space by J. Matkowski [2], in which the classical Banach metric inequality

$$d\big(F(x), F(y)\big) \le k \cdot d(x, y) \qquad (*)$$

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was replaced by the inequality $d(F(x), F(y)) \leq \varphi(d(x, y))$ with a nondecreasing function $\varphi(t)$ such that $\varphi^n(t) \to 0$ for any $t > 0$. Let us also mention the result [1, Ch. 1, §1, (5.3)] on general contractions in a complete metric space, where the coefficient k in $(*)$ is replaced by a function $\alpha(x, y)$ with some special properties. In addition, the result by J. Weissinger [5] (see also [1, Ch. 1, §1, (A.5)]) should be mentioned where Banach constant k is replaced by terms of a positive convergent series $\sum_{n=1}^{\infty} a_n$. More exactly, instead of $(*)$, the following inequality was suggested: $d(F^n(x), F^n(y)) \leq a_n \cdot d(x, y)$. In such a way, the geometrical progression with the common ratio k arising in the iteration process in Banach contraction principle was replaced by an arbitrary positive convergent series. A further generalization of Banach principle was obtained by F.E. Browder [6] (see also [1, Ch. 1, §1, (B.2)]) in his famous fixed point theorem, also in a complete metric space, where he considered φ-contractive mapping, that is he used the inequality $d(F(x), F(y)) \leq \varphi(d(x, y))$, with a nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions: (i) $\varphi(t) < t$ for any $t > 0$, (ii) φ is right continuous. The survey of J. Jachymski [7] is devoted to this remarkable Browder theorem, some its generalizations and open questions around it. Of course, we should also mention the well-known Nadler theorem [8], which generalizes Banach contraction principle to the case of multivalued mappings.

All the listed results as well as many others are devoted to the fixed point existence and uniqueness problems. As for analogous coincidence problems, when n $(n \geq 2)$ mappings transform one metric space to another, it is not always possible to reduce them to the fixed point problems. Therefore, different approximation schemes are needed for solving coincidence problems.

To solve some versions of coincidence problems, in this paper we use auxiliary functionals with special properties. In particular, we take advantage of so called functionals strictly subordinated to convergent series (it should be noted that in [15] such functionals were named strictly subjected to convergent series) and functionals compatible with convergent series, which were introduced by the author (see [15,14]). On the one hand, functionals strictly subordinated to convergent series represent some generalizations of (α, β)-search, generally (α, β)-search and almost exactly (α, β)-search functionals $(0 < \beta < \alpha)$ (see [10–12,15,14] for definitions, examples and comparisons). On the other hand, functionals strictly subordinated to convergent series are a particular case of functionals subordinated to convergent series (see [17,18] for the definition and some applications).

It should be noted that the idea of using a majorizing convergent series in the approximation process in some different special form was exploited in [16] with regard to the fixed point existence and approximation problem in the case of multivalued contractions.

Theorems 1–4 presented here were announced in [14]. The proofs of Theorems 1, 2 were given in [15].

Let (X, ρ) be a metric space, $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$ be the set of nonnegative real numbers, and let $P(A)$ stand for the totality of all nonempty subsets of a set A. Let also the following convergent series with monotonically decreasing positive terms be fixed.

$$\sum_{j=1}^{\infty} c_j < \infty, \quad 0 < c_{n+1} < c_n, \ n \in \mathbb{N}. \tag{1}$$

Denote the sum of the residual series of (1) by $S_k = \sum_{j=k}^{\infty} c_n$, $k \in \mathbb{N}$.

Definition 1. ([15,14]) A multivalued nonnegative functional $\varphi : X \to P(\mathbb{R}_+)$ is said to be *strictly subordinated to series* (1) on a metric space (X, ρ), if the following conditions are fulfilled for its graph $Gr(\varphi)$:

1) if for a pair $(x, t) \in Gr(\varphi)$ it is true that $t > c_1$, then there exist a pair $(x', t') \in Gr(\varphi)$ and a number k, $k \in \mathbb{N}$, such that $\rho(x, x') \leq t$ and $t' \leq c_k$;

2) if for a pair $(x, t) \in Gr(\varphi)$ it is true that $t \leq c_k$ for some $k \in \mathbb{N}$, then there exists a pair $(x', t') \in Gr(\varphi)$ such that $\rho(x, x') \leq t$ and $t' \leq \frac{t}{c_k} c_{k+1}$. □