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On isometric embeddings of separable metric spaces

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ABSTRACT

In this paper, we consider spaces having the so-called property of f-distances, where f is a positive decreasing function defined on ω such that $f(n) \leq \frac{1}{2^n}$. It is proved that for well-known classes $\mathbb{S}$ of separable metric spaces (in [2] they are called isometrically ω-saturated classes of spaces) the following is true: for a given collection $\mathbf{S}$ of elements of $\mathbb{S}$ with the property of f-distances, there exists an element of $\mathbb{S}$ with the property of g-distances containing isometrically each element of $\mathbf{S}$, where g is the function on ω for which $g(n) = f(n+2)$, $n \in \omega$.

1. Introduction

We recall some well-known classes of separable metric spaces having isometrically universal elements. In [12] (see also [13]) it is constructed a space which is isometrically universal in the class of all separable metric spaces. In [1] it is proved that the space C[0, 1] of all continuous functions on the segment [0, 1] with the metric of uniform convergence is also such a space. In [2] isometrically universal elements are constructed in the following classes of spaces.

- $\mathbb{S}(1)$: The class of all separable metric spaces.
- $\mathbb{S}(2)$: The class of all separable metric spaces of dimension $\leq n \in \omega$.
- $\mathbb{S}(3)$: The class of all separable metric countable-dimensional spaces.
- $\mathbb{S}(4)$: The class of all separable metric strongly countable-dimensional spaces.
- $\mathbb{S}(5)$: The class of all separable metric locally finite-dimensional spaces.
- $\mathbb{S}(6)$: The class of all separable metric spaces of transfinite dimension ind less than or equal to a countable ordinal α.
- $\mathbb{S}(7)$: The class of all separable metric spaces of metric dimension $\leq m \in \omega$.
- $\mathbb{S}(8)$: The class of all separable metric spaces of metric dimension $\leq m \in \omega$ and dimension ind $\leq n \in \omega$.

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Moreover, in [2] it is proved that the classes $\mathbb{S}(1)$–$\mathbb{S}(8)$ are isometrically ω-saturated classes of spaces. (About topologically universal elements in the classes $\mathbb{S}(1)$–$\mathbb{S}(6)$ see, respectively, [11,7,6,9,10,5].)

In [8] families $\mathbf{F}$ of metric spaces with the following properties are considered: (a) the diameter of each element of $\mathbf{F}$ is less than or equal to a fixed number $\mathbf{d}$ and (b) for every $k \in \omega$ there exists a fixed natural number $\mathbf{n}(k)$ such that each element of $\mathbf{F}$ has a finite $\frac{1}{k}$-net the number of elements of which is less than or equal to $\mathbf{n}(k)$. The following result is proved: the conditions (a) and (b) are sufficient (and obviously necessary) for the existence of a totally bounded metric space containing isometrically all elements of the family $\mathbf{F}$. In [2] a collection of spaces satisfying properties (a) and (b) is called *uniform* and it is proved that if $\mathbf{F}$ is a uniform collection of elements of a isometrically ω-saturated class $\mathbb{S}$ of spaces, then there exists a totally bounded element of $\mathbb{S}$ containing isometrically all elements of $\mathbf{F}$. In particular, $\mathbb{S}$ may coincides with one of the classes $\mathbb{S}(i)$, $i \in \{1, \cdots, 8\}$.

In the present paper we introduce another kind of a "uniform" collection of metric spaces. As the above, for any such uniform subcollection $\mathbf{S}$ of the class $\mathbb{S}(i)$, $i \in \{1, ..., 8\}$, there exists an element of $\mathbb{S}(i)$ containing isometrically all elements of $\mathbf{S}$ and which is very "close" to this subcollection. The metric property which is used for the definition of this kind of "uniformity" appeared in the papers [3] and [4], where it is proved that there exists a separable complete metric space of dimension (in the sense of ind) $n \in \omega^+$ containing isometrically all compact metric spaces of dimension n.

Let f be a positive decreasing function defined on ω such that $f(n) \leq \frac{1}{2^n}$, $n \in \omega$. It is said that a separable metric space X has the property of *f-distances* if it has a base whose elements have the property of *f-distances* (see Section 2 for the definition). A collection $\mathbf{S}$ of separable metric spaces is said to be *f-uniform* if each element of $\mathbf{S}$ has the property of f-distances. The main result is Theorem 3.2 which is proved using the method of the papers [3] and [4]. A corollary of this theorem is the following result: Let $\mathbf{S}$ be an f-uniform family consisting of elements of $\mathbb{S}(i)$, $i \in \{1, \cdots, 8\}$. Then, there exists an element of $\mathbb{S}(i)$ having the property of g-distances, where g is the function on ω for which $g(n) = f(n + 2)$, $n \in \omega$, containing isometrically each element of $\mathbf{S}$. The class $\mathbb{S}(i)$ may be replaced by any isometrically ω-saturated class.

2. Preliminaries

2.1. Notation

All considered spaces are assumed to be separable metric. The metric of a space X is denoted by ρ_X. For every subset Q of a space X, including the empty set, we assume that $\rho_X(\emptyset, Q) = \infty > 0$. Also, we denote by $Cl_X(Q)$, $Int_X(Q)$, $Bd_X(Q)$, and $\mathrm{Diam}(Q)$ its closure, interior, boundary, and diameter, respectively. For every $\varepsilon > 0$ we put

$$O^X_\varepsilon(Q) = \{x \in X : \rho_X(x, Q) < \varepsilon\}.$$

By ω we denote the set of non-negative integers. Each element $n \in \omega \setminus \{0\}$ is identified with the set $\{0, \cdots, n-1\}$. The element 0 is identified with the empty set. Therefore, for two elements $n, m \in \omega$, the relations $n \in m + 1$, $n \subset m$, and $n \leq m$ are equivalent.

2.2. The metric space $\mathrm{T}(\mathbf{M}, \mathrm{R}, \mathbf{P})$

(See [3]. For the notions that are not determined here, see [2, Chapters 1 and 9].) Below, for an indexed collection $\mathbf{S}$ of spaces we give briefly the construction of a compatible metric ρ_{T} on a topological space $\mathrm{T}(\mathbf{M}, \mathrm{R})$, where

$$\mathbf{M} \equiv \{\{U^X_\delta : \delta \in \tau\} : X \in \mathbf{S}\} \quad \text{and} \quad \mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$$