Topology and its Applications 179 (2015) 99–110

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Universal regular and completely regular frames

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ARTICLE INFO

Article history:
Received 1 April 2014
Accepted 9 July 2014
Available online 26 September 2014

MSC:
06A06
06D22
54A05

Keywords:
Lattice
Frame
Universal frame

ABSTRACT

In this paper we prove that in the classes of all regular frames and all completely regular frames there are universal elements giving an affirmative answer to Problems 1.4 (1, 2) of [1]. To obtain these results we simplify the method of construction of universal elements in the class of all frames given in [1] and then use it for the considered classes.

1. Introduction and preliminaries

1.1. Introduction

For a given class of frames the Universality Problem consists in the finding of an element of this class (if it exists), called universal element, with the property that there exists a homomorphism onto of this universal element to each fixed element of the class. In contrast to the Theory of Topological Spaces, where the Universality Problems are considered from the first steps of the General Topology, in the Theory of Frames, the Universality Problems are in the beginning.

In [1] the existence of a universal element in the class of all frames of weight less than or equal to a fixed cardinal is proved. For this the original method of construction of universal elements of classes of spaces given in [2] is used. We note that from the existence of universal elements in a class of frames the existence of universal elements in a greater or in a smaller class of frames does not follow. Thus, in [1] the problems of the existence of universal elements in the class of all regular frames or in the class of all completely regular frames of weight less than or equal to a given cardinal are posed (Problems 1.4 (1, 2)). In the present paper we give affirmative answers to these problems.

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http://dx.doi.org/10.1016/j.topol.2014.08.020

1.2. Definitions and notation

An *ordinal number* is the set of smaller ordinal numbers, and a *cardinal number* is an initial ordinal number. By ω we denote the least infinite cardinal and by τ a fixed infinite cardinal. The elements of τ will be denoted by δ, ε, η, ζ and ξ. The letters θ, ϑ and φ are used for the notation of mappings of indices. Elements of ω will be denoted by k, m and p. For a set X we denote by $\mathcal{F}(X)$ the set of all non-empty finite subsets of X. The set $\mathcal{F}(\tau)$ is denoted simply by $\mathcal{F}$.

Each mapping f of a set κ onto a set A is called a κ-*indexation* of A and will be denoted by $A = \{a_\delta : \delta \in \kappa\}$ where $a_\delta = f(\delta)$, $\delta \in \kappa$. A τ-indexation of a set A will be called simply an *indexation*.

Recall that a *frame* is a complete lattice L in which

$$x \wedge \sup S = \sup\{x \wedge y \colon y \in S\}$$

for any $x \in L$ and any $S \subseteq L$. Our notations shall be fairly standard from Picado and Pultr [3]. In particular, we denote the top element and the bottom element of L by 1_L and 0_L respectively. For two elements x and y of a frame L we put

$$y^* = \sup\{z \in L \colon z \wedge y = 0_L\}$$

and write $y \prec x$ iff $y^* \vee x = 1_L$. Also we write $y \prec\prec x$ iff there is a system

$$\{c_r \in L : r \in \mathbb{Q} \cap [0,1]\},$$

where $\mathbb{Q}$ is the rational numbers, such that $c_0 = y$, $c_1 = x$, and $c_{r_1} \prec c_{r_2}$ whenever $r_1 < r_2$. A frame L is *regular* if for each $x \in L$,

$$x = \sup\{y \in L : y \prec x\}$$

and *completely regular* if for each $x \in L$,

$$x = \sup\{y \in L : y \prec\prec x\}.$$

A frame *homomorphism* is a map between frames which preserves finite infima, including the top element, and arbitrary suprema, including the bottom element.

Let $\mathbb{L}$ be a class of (non-empty) frames. We say that a frame T is *universal* (see [1]) in this class if (a) $\mathrm{T} \in \mathbb{L}$ and (b) for every $L \in \mathbb{L}$ there exists a homomorphism of T onto L. If only the second condition is satisfied, then T is called a *containing* frame for $\mathbb{L}$.

A subset B of a frame L is called a *base* of L if each element of L is the supremum of a subset of B. The *weight* of a frame L is the minimal cardinal κ for which there exists a base B of L of cardinality κ.

Let B be a base of a frame L and $x, y \in L$. We denote by y^*_B the supremum of all elements $z \in B$ such that $y \wedge z = 0_L$. We shall write $y \prec_B x$ iff $y^*_B \vee x = 1_L$. The correspondence $y \to y^*_B$ will be called B-*star operator* on L. We shall say that the base B is *closed under taking the* B-*star operator* if for every $y \in B$ we have $y^* \in B$. The base B is said to be *regular* if for each element $a \in B$ we have

$$a = \sup\{z \in B : z \prec_B a\}.$$

Also we shall write $y \prec\prec_B x$ iff there is a system

$$\{c_r \in B \colon r \in \mathbb{Q} \cap [0,1]\} \tag{1.1}$$