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On covariant functors in the category of compact Hausdorff spaces

A.V. Ivanov *, K.V. Matyushichev

A R T I C L E I N F O

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A B S T R A C T

We investigate covariant functors in the category of compact Hausdorff spaces and obtain a series of results allowing to construct normal and seminormal functors of finite degree with desired properties. In particular, it turns out that any functor of finite degree is a factor-functor of some monomorphic functor of the same degree. Under **CH**, a compact discretely generated space Z is constructed such that $\mathcal{F}(Z)$ is not discretely generated for any seminormal functor $\mathcal{F}$ of finite degree.

1. Introduction

The notion of a normal functor in the category of compact Hausdorff spaces was first introduced by E.V. Ščepin in [15]. A classical result of Katětov says that if the 3rd power of a compact Hausdorff space is hereditarily normal, then the space is metrizable. In 1987 V.V. Fedorchuk [8] generalized the Katětov theorem to any normal functor of degree ≥ 3. Later, the Fedorchuk theorem was transferred to a wide class of seminormal functors. Finally, under the assumption of **CH** all seminormal functors of finite degree, for which the generalized Katětov theorem holds, were described in [11]. A similar situation arises when one tries to transfer the A.V. Arhangel'skiĭ and A.P. Kombarov theorem [1] (which says that if the space $X^2 \setminus \Delta$ is normal (here X is an arbitrary compact Hausdorff space and $\Delta = \{(x, x) : x \in X\}$), then $\chi(X) \leq \omega_0$)

* Corresponding author.

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to normal and seminormal functors other than that of raising to the second power. It turns out that under the assumption of Jensen's Principle ($\diamondsuit$) this theorem does not hold for functors $\exp_n$ (see [13]). At the same time, the A.V. Arhangel'skiĭ and A.P. Kombarov theorem can be generalized to the so called regular functors. Such results require rather subtle investigations of the structure of the functors of finite degree. This article is mainly devoted to the study of properties of such functors.

Let us mention some of the results we obtained.

It is proved that a functor of finite degree is seminormal if and only if this functor is finitely monomorphic, preserves a one-point space, preserves the empty space, and preserves an empty intersection of finite spaces.

It is proved that the Basmanov mapping $\pi_{\mathcal{F}Xn}$ (see [4]) is continuous for any (not necessarily monomorphic) functor of finite degree.

On the basis of the Basmanov theorem (see [3]), normality and seminormality of functors of degree n are characterized in terms of their action in the category n. This makes it possible to construct functors of finite degree with desired properties.

It is proved that any (not necessarily monomorphic) functor of degree n is a factor functor of some monomorphic functor of the same degree.

Some of the results are concerned with interrelations between the properties of finite monomorphicity, preservation of finite intersections, and continuity of functors.

The last section of the article is devoted to the question whether seminormal functors of finite degree preserve the property of being discretely generated. Originally, this question arose (see [5]) for the functor of raising to the second power: is it true for a discretely generated compact Hausdorff space Y that Y^2 is also discretely generated? A counterexample (denoted by X) to the question was constructed in [14] under the assumption of **CH**. It is worthwhile to note that in constructing X the method of resolvents in the sense of V.V. Fedorchuk (see [7]) is applied. Thus, in the terminology of [2] the space X is an F-compact space (a Fedorchuk compact space). It turns out that the space X can serve as a starting point in constructing a discretely generated compact space Z such that $\mathcal{F}(Z)$ is not discretely generated for any seminormal functor $\mathcal{F}$ of finite degree.

2. Definitions and some properties of functors

We do not distinguish between $\subset$ and $\subseteq$. *Comp* denotes the category of all compact Hausdorff spaces and their (continuous) mappings. Only full subcategories of the category *Comp* and only covariant functors between them will be considered. By *Comp** we denote *Comp* without the empty space. For any $X, Y \in$ *Comp* let Y^X be the set of all the continuous mappings $f : X \to Y$. For any $f \in Y^X$ let $\mathrm{Im}(f) = f(X) \subset Y$ be the image of X. Further, $Comp_0 = \{X \in Comp : \dim(X) = 0\}$. If $A \subset X$ then $i_{A,X}$ denotes the inclusion mapping $A \to X$ which sends each point of A to itself (in X).

A functor $\mathcal{F}$ is *monomorphic* [15] if for every embedding $i : Y \to X$ the mapping $\mathcal{F}(i) : \mathcal{F}(Y) \to \mathcal{F}(X)$ is an embedding, too. For a monomorphic functor $\mathcal{F}$ and a closed subset $A \subset X$ the space $\mathcal{F}(A)$ is naturally identified with the subspace $\mathcal{F}(i_{A,X})(\mathcal{F}(A))$ of the space $\mathcal{F}(X)$. Observe that any functor maps homeomorphisms to homeomorphisms.

Remark 2.1. Let A be a closed subset of a compact space X, for which there exists a retraction $r : X \to A$. Then $r \circ i_{A,X} = id_A$. Consequently, the mapping $\mathcal{F}(i_{A,X})$ is an embedding and $\mathcal{F}(A)$ can be considered as a subspace of $\mathcal{F}(X)$. One readily observes that the mapping $\mathcal{F}(r) : \mathcal{F}(X) \to \mathcal{F}(A)$ is a retraction, too. Let us note that all the above statements hold for every finite subset A of any zero-dimensional compact space X.

They say that a functor $\mathcal{F}$ preserves intersections [15] if for any X and any family $\{A_\alpha\}$ of closed subsets $A_\alpha \subset X$ the following equality holds: