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Computation of inductive dimensions of product of compacta [☆]

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ABSTRACT

Using dimension-like invariant base dimension I of a space by a normal base, the inductive dimensions of the product of compacta in Filippov's example are computed.

1. Introduction and preliminaries

There are several reasons to study dimension-like invariants: to get more information about spaces; to extend the class of spaces under investigation; to evaluate the classical dimensions, etc. M. Charalambous, A. Chigogidze, V. Filippov, J. Aarts, T. Nishiura, B. Pasynkov, S. Iliadis are a few to be mentioned in connection with these themes of research. The last works of V.V. Fedorchuk are connected with the problem of finding unifying approaches to the study of set-theoretical methods in the classical dimension theory and algebraic methods in the investigation of cohomological dimensions.

Useful information about dimension-like invariants of the covering dimension type can be found in [10] and of the type Ind in [9,4]. It is also worth to mention papers [2] and [12]. In the first M. Charalambous used uniform dimensions to obtain the general form of the subset theorem for dimension dim. In the second relative dimensions are used for evaluation of dimension dim of partial products.

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In this paper we demonstrate how base dimension I can be used to compute the classical dimension Ind of products.

B. Pasynkov [15] established the finite-dimensionality of the product of two finite-dimensional compacta. Moreover, the recursion relation for the function that estimate dimension of the product by dimensions of its factors was given there (see also [16] in this connection). V. Filippov [8] constructed an example of compact spaces X, Y such that

$$\operatorname{Ind} X = 1, \qquad \operatorname{Ind} Y = 2 \quad \text{and} \quad \operatorname{Ind} X \times Y \geq \operatorname{ind} X \times Y > 3.$$

By Pasynkov's estimation $\operatorname{Ind} X \times Y \leq 7$, and $\operatorname{ind} X \times Y \leq 4$ by [5].

We shall show that $\operatorname{Ind} X \times Y = \operatorname{ind} X \times Y = 4$ in this case. It is worth to note that the authors know only two examples of compacta when the sum of large inductive dimensions of factors is less than the large inductive dimension of the product. Another one is a modification of Filippov's example by D. Malykhin [14]. In his example the one-dimensional factor is linearly ordered.

The proof uses estimates of dimension Ind by dimension I introduced by S. Iliadis [11] and the product theorem for I which is motivated by I. Lifanov's product theorem [13]:

$$\operatorname{Ind}(X_1 \times \ldots \times X_k) \leq \operatorname{Ind} X_1 + \ldots + \operatorname{Ind} X_k$$

if the weak form of the finite sum theorem is fulfilled in the factors. The following natural question arises.

Question 1.1. Does every compactum that is finite-dimensional in the sense of Ind have a normal base such that its respective base dimension I is also finite, and the (weak) finite sum theorem for I is fulfilled?

On every normal (even Tychonoff) space there exists a normal base (the family of all zero sets) with respect to which the finite sum theorem is fulfilled (for details see, for example, [9]). However, it is shown in [3] that there are finite dimensional compacta in the sense of Ind whose dimension Ind_0 (base dimension I by the normal base of all zero sets) is infinite. Dimension Ind_0 was introduced independently by M. Charalambous and V. Filippov (for details see, for example, [9]).

All spaces are assumed to be Tychonoff, maps are continuous and notations, terminology and designations are from [6]. Homeomorphic spaces X and Y are denoted as $X \cong Y$, $\mathbb{N}$ is the set of natural numbers, $W(\alpha)$ is the space of all ordinals less than α with order topology. By a neighborhood we always understand an open neighborhood; $\operatorname{cl} A$ and $\operatorname{int} A$ are the closure and interior of a subset A of a space X respectively.

2. Normal base, base dimension I

2.1. Normal bases

Let X be a space. A family $\mathcal{F}$ of closed subsets of X is called *a base for the closed subsets of* X if the family $\mathcal{F}^c = \{X \setminus F : F \in \mathcal{F}\}$ is a base for the open subsets of X.

A base $\mathcal{F}$ for the closed subsets of X is said to be *normal* if:

(i) $\mathcal{F}$ is a *ring of sets on the set* X: $\mathcal{F}$ is closed under finite unions and finite intersections;
(ii) $\mathcal{F}$ is *disjunctive*: $\forall G \in \mathcal{F}$ and $\forall x \notin G$ there exists $F \in \mathcal{F}$ such that $x \in F$ and $G \cap F = \emptyset$;
(iii) $\mathcal{F}$ is *base-normal*: for a given pair (F_1, F_2) of disjoint elements of $\mathcal{F}$ there exists a pair (G_1, G_2) of elements of $\mathcal{F}$ such that $F_1 \cap G_2 = \emptyset$, $F_2 \cap G_1 = \emptyset$, and $G_1 \cup G_2 = X$, or, equivalently, there exist disjoint elements O_1, O_2 of $\mathcal{F}^c$ such that $F_1 \subset O_1$ and $F_2 \subset O_2$. (The pair (G_1, G_2) is called a *screening* of the pair (F_1, F_2). Note that the sets $G_1 \cap G_2$ and $L = X \setminus (O_1 \cup O_2)$ are partitions between F_1 and F_2 in X, which are elements of $\mathcal{F}$. Such partitions will be called $\mathcal{F}$*-partitions*.)

We denote $(X, \mathcal{F}) \cong (Y, \mathcal{F}')$ if there exists a homeomorphism $h: X \to Y$ such that $\{h(F) : F \in \mathcal{F}\} = \mathcal{F}'$.