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Multivalued fixed point results in cone metric spaces

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A R T I C L E   I N F O

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To the memory of Vitaly V. Fedorchuk

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A B S T R A C T

In this paper we extend main fixed point results of Kikkawa and Suzuki (2008) [19] and Mot and Petruşel (2009) [21] for the case of cone metric spaces without assumption of normality on cone. We also support our results by a nontrivial example and establish a homotopy theorem as an application.

1. Introduction

The Banach contraction theorem plays a fundamental role in fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer. For more details in fixed point theory we refer the reader to [13,14,22,24]. On the other hand, it cannot describe the metric completeness. To overcome this issue, in [26] Suzuki established a fixed point theorem which generalized the Banach contraction theorem and characterized the metric completeness. In [19] Kikkawa and Suzuki extended the main theorem of [26] for the case of multivalued mappings as:

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Theorem 1.1. *([19]) Define a strictly decreasing function* φ *from* $[0,1)$ *onto* $(\frac{1}{2},1]$ *by* $\varphi(t)=\frac{1}{1+t}$*. Let* (X,d) *be a complete metric space and* $F: X \to \mathsf{CB}(X)$ *be a multi-valued mapping. Assume that there exists* $a \in [0,1)$ *such that*

$$\varphi(a)d(x,Fx) \leq d(x,y) \quad \Longrightarrow \quad H(Fx,Fy) \leq ad(x,y)$$

for all $x,y \in X$*. Then there exists* $u \in X$ *such that* $u \in Fu$*.*

Here, $\mathsf{CB}(X)$ denotes the family of all nonempty closed bounded subsets of X.

In [21] Mot and Petruşel, presented the result of [19] for locally contractive mappings, in the context of complete metric spaces.

Theorem 1.2. *([21]) Define a strictly decreasing function* φ *from* $[0,1)$ *onto* $(\frac{1}{2},1]$ *by* $\varphi(t)=\frac{1}{1+t}$*. Let* (X,d) *be a complete metric space,* $x_0 \in X$*, and* $F: X \to \mathsf{CB}(X)$ *a multi-valued mapping. Assume that there exist* $a \in [0,1)$ *and* $r > 0$ *such that*

$$\varphi(a)d(x,Fx) \leq d(x,y) \quad \Longrightarrow \quad H(Fx,Fy) \leq ad(x,y)$$

for all $x,y \in \overline{B}(x_0,r)$ *and*

$$d(x_0,Fx_0) \leq (1-a)r.$$

Then there exists $u \in \overline{B}(x_0,r)$ *such that* $u \in Fu$*.*

In 2007, Huang et al. [17] introduced cone metric spaces with normal cone, as a generalization of metric spaces.[1] Rezapour et al. [23] presented the results of [17] for the case of cone metric space without normality in cone. In recent years many authors workout on fixed point theory in cone metric spaces and other generalizations of metric spaces (see [1–11,15,16,18]). Cho et al. [12] invented the Hausdorff distance function on cone metric spaces and generalized the result of [20] for multivalued mappings.

In this article we achieved the generalization of the results of [19] and [21] by using Hausdorff distance function introduced by [12] for a cone metric. As an application a homotopy result for locally contractive mappings in cone metric space have been established.

We also have given an example to support our main theorem.

2. Preliminaries

Let $\mathbb{E}$ be a real Banach space and P be a subset of $\mathbb{E}$. By θ we denote the zero element of $\mathbb{E}$. The subset P is called a *cone* if and only if:

(i) P is closed, nonempty, and $P \neq \{\theta\}$;
(ii) $a,b \in \mathbb{R}$, $a,b \geq 0$, $x,y \in P \Rightarrow ax+by \in P$;
(iii) $P \cap (-P) = \{\theta\}$.

For a given cone $P \subseteq \mathbb{E}$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ (or $y \succcurlyeq x$) if and only if $y-x \in P$; $x \prec y$ stands for $x \preceq y$ and $x \neq y$, while $x \ll y$ (or $y \gg x$) stands for $y-x \in \mathrm{int}(P)$, where $\mathrm{int}(P)$ denotes the interior of P. The cone P is said to be *solid* if it has nonempty interior.

In the sequel we use P as a non-normal solid cone of a real Banach space $\mathbb{E}$.

[1] In fact, cone metric spaces have been introduced in 1934 by Serbian mathematician Dj. Kurepa (espaces pseudo-distanciés), and then redefined and investigated in many papers under various names.