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Metrizable DH-spaces of the first category

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ABSTRACT

We show that if a separable space X contains an open subset which is of the first category in itself and is not a λ-space, then X has $\mathfrak{c}$ many types of countable dense subsets. We introduce Λ-spaces as a generalization of the λ-spaces for non-separable case and consider properties of these spaces. In particular, we prove that if X is a non-σ-discrete h-homogeneous Λ-space, then X is densely homogeneous and $X \setminus A$ is homeomorphic to X for every σ-discrete subset $A \subset X$.

All spaces under discussion are metrizable.

In the present time we see the rapid development of the theory of CDH-spaces and their near generalizations. For example, K. Kunen, A. Medini, and L. Zdomskyy [9, Theorem 16] proved that if a separable metrizable non-Baire space X has the perfect set property for open sets, then X has $\mathfrak{c}$ types of countable dense subsets. We will show (see Corollary 2) that a similar result remains valid if we replace "X has the PSP(open)" by the weaker condition "X is not a λ-space".

We introduce the Λ-spaces as a generalization of the λ-spaces for non-separable spaces and consider their properties. Theorem 4 shows how a Λ-space can be converted into an h-homogeneous Λ-space of arbitrary weight. Theorem 5 improves the result due to R. Hernández-Gutiérrez, M. Hrušák, and J. van Mill [5, Proposition 4.9] concerning CDH-property of h-homogeneous λ-spaces. An internal characteristic of h-homogeneous DH-spaces of the first category is given by Theorem 6.

In the paper we are not dealing with the set-theoretic methods; only topological methods are applied. In particular, a purely topological way in obtaining a CDH λ-space of size $\aleph_1$ is found, see Remark 4.

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1. DH-Spaces

For all undefined terms and notations see [2].

$X \approx Y$ means that X and Y are homeomorphic spaces. A separable topological space X is *countable dense homogeneous* (briefly, CDH) if, given any two countable dense subsets A and B of X, there is a homeomorphism $h: X \to X$ such that $h(A) = B$. The *type* of a countable dense subset D of a separable space X is the family $\{h(D): h$ is a homeomorphism of $X\}$. So a separable space is CDH if and only if it has exactly one type of countable dense subsets.

A metrizable space X is *densely homogeneous* (briefly, DH) provided that if A and B are two σ-discrete dense subsets of X, then there is a homeomorphism $h: X \to X$ such that $h(A) = B$. Clearly, if X is a separable space, then X is CDH $\Leftrightarrow$ X is DH.

A space X is called a space *of the first category* (or *meager*) if it can be represented as a countable union of nowhere dense subsets.

Lemma 1 was obtained by the author [15, Theorem 3]. Independently it was proved for separable spaces by B. Fitzpatrick Jr. and H.-X. Zhou [4].

Lemma 1. *For a metric space X the following are equivalent:*

1) *the space X is of the first category,*
2) *X contains a σ-discrete dense set of type G_δ without isolated points.*

Recall that a separable space in which every countable set is a G_δ-set is called a λ-*space*. This notion is due to Kuratowski [10]. Likewise, a space in which every σ-discrete set is a G_δ-set will be called a Λ-*space*. Of course, a separable Λ-space is a λ-space. From Lemma 1 it follows that every metrizable Λ-space without isolated points is of the first category in itself. One can check that if a space X contains a copy of the Cantor set 2^ω, then X is not a Λ-space.

For a remarkable survey of what is known to date about λ-spaces, see [5].

The following statement is similar to [4, Theorem 3.4].

Theorem 1. *Every DH-space X of the first category is a Λ-space.*

Proof. Take a σ-discrete subset A of X. By Lemma 1, there is a σ-discrete dense G_δ-set $B \subset X$. Clearly, $A \cup B$ is a σ-discrete dense subset of X. Then $A \cup B$ is a G_δ-set in X because $A \cup B = h(B)$ for some homeomorphism $h: X \to X$. Since A is a G_δ-set in $A \cup B$, A is a G_δ-set in X. $\square$

Corollary 1. *Let X be a space of the first category. If X is not a Λ-space, then X is not DH.*

Proof. If X were a DH-space, X would be a Λ-space by Theorem 1. A contradiction. $\square$

We shall now show the following improvement on Corollary 1. To obtain this result we shall use ideas from the proof of [6, Theorem 4.5].

Theorem 2. *Suppose a space X has an open subset which is of the first category in itself and is not a Λ-space. Then X has at least $\mathfrak{c}$ types of σ-discrete dense subsets.*

Proof. The set $V = \bigcup\{U: U$ is an open set of the first category in $X\}$ is the largest open set of the first category in X. The set $Y = X \setminus \bigcup\{U: U$ is a Λ-space and U is open in $X\}$ is closed in X. One can check that a point $y \in Y \Leftrightarrow$ every neighborhood of y is not a Λ-space. Under the conditions of the theorem, $V \cap Y \neq \emptyset$. We shall consider two possibilities.