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Fiberwise contraction mappings principle

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ABSTRACT

Banach's contraction mappings principle is extended over metric mappings.

Below a space means a topological space.

In this paper, Banach's contraction mappings principle will be extended from the case of metric spaces over the case of metric mappings.

Recall (see [1]) that a *metric on a mapping* f of a set X to a space (Z, θ) is a pseudometric ρ on X such that it is a metric on every fiber $f^{-1}z$ of f, $z \in Z$. *The topology* $\tau(f, \rho)$ *on* f *generated by the metric* ρ *on* f is the topology on X with the base $\tau_\rho \wedge f^{-1}\theta = \{U \cap f^{-1}O : U \in \tau_\rho, O \in \theta\}$, where τ_ρ is the topology on X generated by the pseudometric ρ.

A pair (f, ρ) consisting of a mapping f of a set to a space and of a metric ρ on f is called a *metric mapping*. Evidently, for every metric mapping (f, ρ) the mapping $f : (X, \tau(f, \rho)) \to Z$ is continuous. (Usually for any metric mapping $(f, \rho) : X \to Z$, we shall consider X with the topology $\tau(f, \rho)$.) A metric mapping (f, ρ) is called *fiberwise complete* if ρ is a complete metric on every fiber of f.

For a continuous mapping $f : X \to Z$, a continuous mapping $A : X \to X$ is called a *map-morphism* $A : f \to f$ (of f to f) if $f \circ A = f$ (hence $A(f^{-1}z) \subset f^{-1}z$ for any $z \in Z$).

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For a metric mapping $f : X \to Z$, a map-morphism $A : f \to f$ is called *β-contracting* (≡ a *β-contraction*) for a real-valued function $\beta(z)$, $z \in Z$, if for any $z \in Z$ and any $x, x' \in f^{-1}z$,

$$0 \le \beta(z) < 1 \quad \text{and} \quad \rho\big(Ax, Ax'\big) \le \beta(z) \cdot \rho\big(x, x'\big).$$

Thus *β-contracting map-morphism $A : f \to f$ is a $\beta(z)$-contracting mapping of the fiber $f^{-1}z$ for any $z \in Z$.*

Definition 1. Let we have a continuous mapping $f : X \to Z$. A map-morphism $r : X \to X$ will be called a *fiberwise retraction* (or an *f-retraction*) of X onto $R = rX$ if $r(x) = x$ for any $x \in R$. In this situation, R will be called a *fiberwise retract* (or an *f-retract*) of X.

Definition 2. For a continuous mapping $f : X \to Z$, a (not necessary continuous) mapping $s : Z \to X$ will be called a *retract section* of f if $f \circ s = id_Z$ and sZ is an f-retract of X. (Evidently, s is a one-to-one mapping of Z on sZ.)

For a continuous mapping $f : X \to Z$, a real-valued nonnegative function β on Z will be called *locally strongly 1-bounded* if for any $z \in Z$ there exist a neighbourhood Oz of z and a positive number $\gamma = \gamma(z) < 1$ such that $\beta(z') \le \gamma$ for all $z' \in Oz$.

Theorem 1 *(The weak fiberwise contraction mappings principle). Let we have a fiberwise complete metric onto mapping $f : X \to Z$ and let a map-morphism $A : f \to f$ be β-contracting for a locally strongly 1-bounded function $\beta(z)$, $z \in Z$, then there exists a unique retract section $s : Z \to X$ of f such that sZ consists of all fixed points of the mapping $A : X \to X$ (hence $A \circ s = s$) and*

$$(*)\ \rho\big(x, s(fx)\big) \le \rho(x, Ax) \cdot \frac{1}{1 - \beta(fx)}, \quad x \in X.$$

Proof. It follows from the standard proof of the theorem on contracting mappings of complete metric spaces and from the fiberwise completeness of f (and because A is a map-morphism) that for any $z \in Z$ and any $x \in f^{-1}z$ the sequence $A^n x$, $n = 0, 1, \ldots$, converges to a point $r(x) \in f^{-1}z$, that is a fixed point for A (i.e. $A(r(x)) = r(x)$). Since $A|_{f^{-1}z}$ has only one fixed point, the points $r(x)$ coincide for all $x \in f^{-1}z$. Let $s(z)$ denote the point $r(x)$ for all $x \in f^{-1}z$. Now we have two mappings $r : X \to X$ and $s : Z \to X$. Note that

$$A\big(s(z)\big) = s(z), \quad z \in Z$$

(indeed, $A(s(z)) = A(r(x)) = r(x) = s(z)$ for any $x \in f^{-1}z$);

$$f \circ s = id_Z;$$

$$r = s \circ f$$

and $f|_{sZ}$ is one-to-one (and continuous).

Let us show that r is continuous.

Let $x \in X$, $\varphi(x) = \rho((x = A^0 x), Ax)$ and $z = fx$.

As in the standard proof of the contraction mappings principle, for all $m, n \in \{0, 1, \ldots\}$, $m < n$,

$$\rho\big(A^m x, A^n x\big) \le \varphi(x) \cdot \big(\beta(z)\big)^m \frac{(\beta(z))^{n-m} - 1}{\beta(z) - 1}.$$

Since ρ is a continuous function of the second variable,

$$\rho\big(A^m x, r(x)\big) = \rho\big(A^m x, \big(s(fx) = s(z)\big)\big) \le \varphi(x) \frac{(\beta(z))^m}{1 - \beta(z)}. \tag{1}$$