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On paraconvexity in spaces of summable mappings

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ABSTRACT

A new selection theorem is proved for lower semicontinuous mappings $F : X \to L_M(T; B)$ into an Orlisz spaces of summable mappings. Values $F(x)$ of F in this theorem in general are neither convex nor decomposable, they are unions of two sets which are both convex and decomposable. The key ingredient of the proof is an appropriate estimate of nonconvexity of such union.

1. Introduction

The famous Michael's selection theorem [10] states that every LSC multivalued mapping F of a paracompact domain into a Banach space admits a single-valued continuous selection whenever all values of F are nonvoid, closed and convex. Alternatively, based on tasks arising in the theory of differential inclusions, Fryszkowski [6], Bressan and Colombo [4], Tolstonogov [16], Ageev and Repovs [1] proved an analogous selection theorems for decomposable-valued mappings for various kinds of domains and for ranges which are L_p-spaces. Recall, that a nonvoid set Z of measurable mappings (equivalence classes with respect to a.e. relation) from a probabilistic space (T, Ω, μ) is said to be *decomposable* if for every $f \in Z$, $g \in Z$ and every measurable $A \in \Omega$ the piecewise defined mapping $h(t) = f(t)$, $t \in A$, and $h(t) = g(t)$, $t \notin A$ also belongs to Z.

In this paper we shall prove selection theorems for LSC (and, even for quasi-LSC) mappings whose values are nonconvex, nondecomposable subsets of summable functions.

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So, we say that a pair (C_1, C_2) of closed convex (nonvoid) subsets of a separable Banach space B is a *convex pair* if the union $C = C_1 \cup C_2$ also is convex and $C_1 \setminus C_2 \neq \emptyset$, $C_2 \setminus C_1 \neq \emptyset$. A subset S of a linear space Y of measurable mappings from T to B is said to be *partially convex* if

$$S = \{ f \in Y \mid f(t) \in C_1 \text{ or } f(t) \in C_2 \}$$

for some convex pair (C_1, C_2).

The subsets $S_i = \{f \in Y \mid f(t) \in C_i\}$, $i = 1, 2$ and $\{f \in Y \mid f(t) \in C = C_1 \cup C_2\}$ are convex and decomposable. But the partially convex set $S = S_1 \cup S_2$ can be neither convex nor decomposable. In fact, if $f_1(t) \in C_1$, $f_2(t) \in C_2$, $t \in T$, $2g = f_1 + f_2$, $h|_A = f_1|A$, $h|_{T\setminus A} = f_2|_{T\setminus A}$, then clearly in general $h \notin S$ and $g \notin S$, because all three positions $g(t) \in C_1 \setminus C_2$, $g(t) \in C_1 \cap C_2$, and $g(t) \in C_2 \setminus C_1$ are possible for various $t \in T$.

Below, $M : [0; +\infty) \to [0; +\infty)$ denotes an Orlisz function, i.e. a convex, continuous mapping with $\lim_{u\to 0} \frac{M(u)}{u} = 0$, $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$ and let $L_M = L_M(T, B)$ be the Banach space of all (equivalence classes of) M-summable in the Bochner sense mappings f from a probabilistic space (T, Ω, μ) to B endowed with the standard norm

$$\|f\| = \inf\left\{ r > 0 \;\Big|\; \int_T M\left(\frac{\|f(t)\|_B}{r}\right) d\mu \leq 1 \right\}.$$

Note, for $M(u) = u^p$, $1 \leq p < \infty$ the Banach space $L_M(T, B)$ coincides with the more classical $L_p(T, B)$ endowed with the norm $\|f\| = (\int_T \|f(t)\|_B^p \, d\mu)^{1/p}$. In order to save the assumption of separability which is essential for the theory of Bochner integration, below we shall assume that Orlisz function $M : [0; +\infty) \to [0; +\infty)$ globally satisfies to the Δ_2-condition $M(2u) \leq m \cdot M(u)$ for some $m > 2$ and all u. Then $L_M(T, B)$ is the separable Banach space. For details on Orlisz spaces see [9].

Theorem 1. *A LSC mapping $F : X \to L_M(T; B)$ of a paracompact space X admits a continuous single-valued selection, provided that all values $F(x)$ are partially convex.*

Roughly, we shall find an appropriate uniform estimate for nonconvexity of values $F(x)$ and after this apply the so-called paraconvex-valued Michael's selection theorem. So, Theorem 1 is the immediate corollary of the two following theorems.

Theorem 2. *Let (C_1, C_2) be a convex pair in a separable Banach space B. Then the set $L_M(T; C_1) \cup L_M(T; C_2)$ is α-paraconvex subset of $L_M(T; B)$ for some $\alpha < 1$.*

Theorem 3. *([11]) A LSC multivalued mapping $F : X \to Y$ of a paracompact space X into a Banach space admits a continuous single-valued selection, provided that for some $0 \leq \alpha < 1$ every value $F(x)$, $x \in X$ is α-paraconvex subset of Y.*

In the special case when $B = H$, $L_M = L_2$ are Hilbert spaces the following version of Theorem 2 for pairs (C_1, C_2) with nonconvex union $C_1 \cup C_2$ can be proved.

Theorem 4. *Let C_1, C_2 be convex closed subsets of a Hilbert space H with nonempty $C_1 \cap C_2$ and with the angle between C_1 and C_2 is more than or equals β for some $\beta > 0$. Then the union $L_2(T; C_1) \cup L_2(T; C_2)$ is α-paraconvex subset of $L_2(T; H)$ for some $\alpha < 1$.*

As a corollary, one can formulate a selection theorem for *LSC* mappings into the Hilbert space $L_2(T; H)$ with values which are bouquets of such types. The advantage of the Hilbert space here is that "nonconvexity"