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Infinite-dimensional hyperspaces of convex bodies of constant width

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A B S T R A C T

There exists a natural embedding of the hyperspace of compact convex bodies of constant width in $\mathbb{R}^n$ into those of $\mathbb{R}^{n+1}$. This allows us to consider the hyperspace of compact convex bodies of constant width in the space $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$. We prove that this hyperspace is homeomorphic to $Q^\infty = \varinjlim Q^n$, where Q denotes the Hilbert cube.

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1. Introduction and preliminaries

We consider the space $\mathbb{R}^n$ endowed with the standard Euclidean metric d generated by the standard inner product $\langle\cdot,\cdot\rangle$.

Given a compact convex body $A \subset \mathbb{R}^n$, its support function $h_A: S^{n-1} \to \mathbb{R}$ is defined by the formula: $h_A(u) = \max\{\langle u, a\rangle \mid a \in A\}$. A compact convex body $A \subset \mathbb{R}^n$ is said to be a body of constant width $r > 0$ if $|h_A(u) - h_A(-u)| = r$, for every unit vector u in $\mathbb{R}^n$. If A is a convex body of constant width r and $a \in \operatorname{bd} A$, we say that $b \in A$ is opposite to a whenever $d(a, b) = r$.

The convex bodies of constant width have many interesting properties and applications. There is a numerous literature devoted to geometry of such convex bodies (see, e.g., survey articles [6,11]).

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It seems natural to consider the convex bodies of constant width in infinite-dimensional spaces. The definition of these bodies can be easily extended over arbitrary Banach spaces (see, e.g., [13]). In this note, we define convex bodies of constant width in an infinite-dimensional space which seems to be one of the closest to the Euclidean spaces, namely, the direct limit of the spaces $\mathbb{R}^n$.

By $\exp X$ we denote the hyperspace (the set of all nonempty compact subsets) of a metric space X. Recall that the Hausdorff metric d_H on $\exp X$ is defined by the formula

$$d_H(A,B) = \max\Big\{\max_{x\in A} d(x,B), \max_{y\in B} d(y,A)\Big\}.$$

The hyperspace of compact convex bodies of constant width is considered in [3] (see also [4]). One of the main results here (Theorem 1.1) states that this hyperspace is a contractible Hilbert cube manifold. This could be compared with the following result of Nadler, Quinn and Stavrakas [14, Theorem 7.3]: the hyperspace of compact convex bodies in $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the punctured Hilbert cube. In [1] it is remarked that the proof of the mentioned result in [4] contains a gap; fortunately, in [1] it is shown that this gap can be easily filled. Moreover, in [1] it is proved that the hyperspace of compact convex bodies of constant width in $\mathbb{R}^n$, $n \geq 2$, is homeomorphic to the product $Q \times \mathbb{R}^{n+1}$, where Q is the Hilbert cube.

The aim of the present note is to prove a counterpart of this result for the hyperspace of convex sets of constant width in the space $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ (see details below). To this end, we first consider a natural embedding of the hyperspace of compact convex bodies of constant width in $\mathbb{R}^n$ into those of $\mathbb{R}^{n+1}$.

Given a nonempty subset $X \subset \mathbb{R}^n$, we define the circumsphere of X as the sphere of the smallest radius containing X. The radius of the circumsphere is called the circumradius of X. It is well known that every convex body of constant width $\leq r$ is of circumradius $\leq r$.

Recall that a convex body in $\mathbb{R}^n$, $n \geq 2$, is said to be strictly convex if its boundary does not contain a line segment.

Let $\bar{B}_r(x)$ (respectively $S_r(x)$) denote the closed ball (respectively sphere) of radius r centered at $x \in X$. Alternatively, a convex body A is of constant width r if $A - A = \bar{B}_r(0)$.

If A is a convex body of constant width $r > 0$, then, for any $x \in \operatorname{bd} A$, there exists $y \in \operatorname{bd} A$ with $d(x,y) = r$; we say that y opposite x.

Let $r > 0$. Following [12] we define $\Omega_r(X) = \bigcap\{\bar{B}_r(x) \mid x \in X\}$ and $\Omega_r^2(X) = \Omega_r(\Omega_r(X))$.

We assume that every $\mathbb{R}^n$ is embedded into $\mathbb{R}^{n+1}$ as follows:

$$(x_1,\dots,x_n) \mapsto (x_1,\dots,x_n,0).$$

By $\mathbb{R}^\infty$ we denote the direct limit of the sequence

$$\mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow \dots$$

Recall that $Q = [0,1]^\omega$ is the Hilbert cube. By Q^∞ we denote the direct limit of the sequence

$$Q \to Q \times \{0\} \hookrightarrow Q \times Q \to Q \times Q \times \{0\} \hookrightarrow Q \times Q \times Q \dots$$

Topological characterizations of the spaces $\mathbb{R}^\infty$ and Q^∞ are obtained by K. Sakai [16]. It is known ([16, Theorem 2.6]; see also [10]) that the Q^∞-manifolds (as well as $\mathbb{R}^\infty$-manifolds) are classified by their homotopy type. In particular, every contractible Q^∞-manifold is homeomorphic to Q^∞.

Recall that a map $f\colon X \to Y$ is called a Z-embedding if the set $f(X)$ is a Z-set in Y, i.e. the identity map 1_Y can be approximated by the maps whose images miss $f(X)$ (see, e.g., [17]).

We will need the following construction from [15]. Let W be any set of constant width λ and $p \notin W$. Let $W_p = W \cap \bar{B}_\lambda(p)$ and let W^* be a set of constant width λ containing $W_p \cup \{p\}$. The set W^* is called

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