Topology and its Applications 174 (2014) 1–13

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Topological and functional analytic properties of the compact-G_δ-open topology on $C(X)$

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ARTICLE INFO

Article history:
Received 30 May 2014
Accepted 5 June 2014
Available online 26 June 2014

MSC:
primary 54C35
secondary 54D99, 54E50, 54E99, 46A11, 46A25

Keywords:
G_δ-set
Submetrizability
Completely metrizable
Čech-complete
Sieve-complete
Partition-complete
Uniform completeness
Separability
Second countable
Nuclear space
Reflexive
Semi-reflexive
Montel space

ABSTRACT

In this paper, we study extensively the compact-G_δ-open topology on the set $C(X)$ of all real-valued continuous functions on a Tychonoff space X. In particular, we study the induced map, submetrizability, completeness properties, countability properties, and nuclear and reflexive properties of the compact-G_δ-open topology on $C(X)$.

1. Introduction

The set $C(X)$ of all continuous real-valued functions on a Tychonoff space X has a number of natural topologies. The idea of topologizing $C(X)$ arose from the notion of convergence of sequences of functions. Also continuous functions and Baire measures on Tychonoff spaces are linked by the process of integration. A number of locally convex topologies on spaces of continuous functions have been studied in order to clarify

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http://dx.doi.org/10.1016/j.topol.2014.06.004
0166-8641/

this relationship. They enable the powerful duality theory of locally convex spaces to be profitably applied to topological measure theory.

Two commonly used topologies on $C(X)$ are the compact-open topology k and point-open topology p. The point-open topology is also known as the topology of pointwise convergence. The study of pointwise convergence of sequences of functions is as old as the calculus. The compact-open topology made its appearance in 1945 in a paper by Ralph H. Fox (see [9]). This topology was shown in [14] to be the proper setting to study sequences of functions which converge uniformly on compact subsets.

It is easily seen that $k = p$ if and only if the compact subsets of X are finite. This condition is quite extreme in nature. So there is a considerable gap between these two topologies. So it is quite natural to fill up this gap by considering some new topologies on $C(X)$ lying between p and k from the viewpoint of topology as well as of topological measure theory. Two such topologies, weak and support-open topologies on $C(X)$ have been studied in [18] and [16]. But it is also interesting to study topologies on $C(X)$ which are weaker than k, but not necessarily stronger than p from the viewpoint of topology as well as topological measure theory.

One such interesting topology, the compact-G_δ-open topology on the space $C(X)$ has been introduced in [11]. The space $C(X)$ with the compact-G_δ-open topology kz is denoted by $C_{kz}(X)$. The subbasic open sets of $C_{kz}(X)$ are of the form

$$[A, V] = \big\{f \in C(X) : f(A) \subseteq V\big\}$$

where $A \in \mathcal{KZ}(X)$, the set of all compact G_δ-subsets of X and V is open in $\mathbb{R}$. In [11], it is shown that for $f \in C(X)$ and $A \in \mathcal{KZ}(X)$, sets of the form $\langle f, A, \varepsilon\rangle = \{g \in C(X) : |f(x) - g(x)| < \varepsilon \text{ for all } x \in A\}$ form a base for the compact-G_δ-open topology on $C(X)$. Consequently, the compact-G_δ-open topology on $C(X)$ is precisely the topology of uniform convergence on compact G_δ-subsets of X. It is also shown that the space $C_{kz}(X)$ is a locally convex space with topology generated by the collection of seminorms $\{p_A : A \in \mathcal{KZ}(X)\}$ on $C(X)$, where

$$p_A(f) = \sup\big\{\big|f(x)\big| : x \in A\big\}.$$

But the space $C_{kz}(X)$ is Hausdorff if and only if $\cup\{K : K \in \mathcal{KZ}(X)\}$ is dense in X. Also $C_p(X) \leq C_{kz}(X)$ if and only if $X = \cup\{K : K \in \mathcal{KZ}(X)\}$, that is, X is of point pseudocountable type and $C_{kz}(X) = C_p(X)$ if and only if every compact G_δ-set in X is finite and $X = \cup\{K : K \in \mathcal{KZ}(X)\}$. Further, $C_{kz}(X) = C_k(X)$ if and only if X is of pseudocountable type, that is, if every compact subset of X is contained in a compact G_δ-set in X.

In [11], in addition to comparing the compact-G_δ-open topology with the point-open and compact-open topologies on $C(X)$, the metrizability, separability and uniform completeness of $C_{kz}(X)$ have been studied. In the present paper, we study several other important properties of the space $C_{kz}(X)$. In particular, we study submetrizability, various kinds of completeness properties, countability properties, and nuclear and reflexive properties of the space $C_{kz}(X)$. We also investigate when $C_{kz}(X)$ is a Montel space.

In Section 2, we first study induced map and sum function in order to study the submetrizability of $C_{kz}(X)$ and show that a number of properties of $C_{kz}(X)$ are equivalent to the submetrizability of $C_{kz}(X)$. In Section 3, we study various kinds of completeness properties of $C_{kz}(X)$ such as complete metrizability, Čech-completeness, local Čech-completeness, sieve-completeness, partition-completeness and quasi-completeness of the space $C_{kz}(X)$. Then in Section 4, we study the countability properties such as $\aleph_0$-boundedness and second countability of the space $C_{kz}(X)$. In particular, we study the $\aleph_0$-boundedness of the topological group $C_{kz}(X)$ in relation to its separability. In the last section of this paper, we study when the locally convex space $C_{kz}(X)$ becomes a nuclear space. Also we investigate when it is reflexive or a Montel space.

Throughout the rest of the paper, we use the following conventions. All spaces are completely regular Hausdorff, that is, Tychonoff (though we may specify that these spaces have additional properties). Unless